Sub-semi-Riemannian geometry on H-type groups

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I_p, I_q – identity matrices $(p \times p)$ and $(q \times q)$ respectively.

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If $p = 1 \Rightarrow \eta$ – Lorentzian metric.

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$$\Rightarrow \quad j(u) = -j(u)^t.$$

Sub-semi-Riemannian *H*-type group

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Define a Lie algebra $\mathcal{G}=U\oplus V$ with a Lie structure on it:

$$[u_1+v_1,u_2+v_2]=[v_1,v_2],\qquad (u_1,u_2\in U,\ \ v_1,v_2\in V)$$

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$$[u_1 + v_1, u_2 + v_2] = [v_1, v_2], \qquad (u_1, u_2 \in U, \;\; v_1, v_2 \in V)$$

$\mathcal{G}$ – 2-step nilpotent Lie algebra with center U,
$\mathcal{G}$ – H-type algebra with nondegenerate product on V.

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G can be considered as a sub-semi-Riemannian manifold (G, V, η).

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$j\colon U \to \mathrm{End}(V)$ – representation of $\mathrm{Cl}(U)$ on V,
$\mathrm{Cl}(U)$ – unital associative algebra which contains and is generated by a vector space U with a quadratic form on it.

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$0 \leqslant m < \rho(n), \quad n = k2^{4r+s} \longmapsto \rho(n) = 8r + 2^s$, k – odd,
$0 \leqslant s \leqslant 3$.

Classification of H-type groups

$m = 0$, n – arbitrary $\Rightarrow \mathcal{G}$ – Euclidean n-dimensional space,
$m = 1$, n – even $\Rightarrow \mathcal{G}$ – Heisenberg algebra,
$m = 3$, $n = 4k \Rightarrow \mathcal{G}$ – Quaternion algebra,
$m = 7$, $n = 8 \Rightarrow \mathcal{G}$ – Octonion algebra.

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j^2-condition

$\forall u_1, u_2 \in U$ with $\langle u_1, u_2 \rangle_U = 0$ $\forall v \in V$ $\exists u_3 \in U$:

$$j(u_1)j(u_2)v = j(u_3)v.$$

Horizontal curve

Absolutely continuous curve $\gamma(t)\colon [0,1] \to G$ such that $\dot\gamma(t) \in V$ $\forall t \in [0,1]$.

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$\forall p \in G$ any tangent vector can be represented as a linear compination of vectors of types: $v(p)$, $[v,w](p) \in T_pG$, v, w – horizontal.

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Chow theorem guarantees that any two points in G can be joined by horizontal curve.

Basic notions

Causal character of vectors v in V

timelike if $\langle v, v\rangle_V < 0$,
spacelike if $\langle v, v\rangle_V > 0$ or $v = 0$,
lightlike if $\langle v, v\rangle_V = 0$ and $v \neq 0$,
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Causal character of horizontal curves

A horizontal curve is called timelike/spacelike/lightlike/nonspacelike if its tangent vector is timelike/spacelike/lightlike/nonspacelike at each point.

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Exponential map

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Left-invariant vector fields

$$V_i = \frac{\partial}{\partial v_i} + \frac{1}{2}\sum_{\alpha=1}^{m} \left(v_j B^{\alpha_{ji}}\right) \frac{\partial}{\partial u_\alpha}, \qquad i = 1, \dots, n,$$

$$U_\alpha = \frac{\partial}{\partial u_\alpha},\ \alpha = 1, \dots, m.$$

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B^α_{ji} represent endomorphisms j_α on V, $\alpha = 1, \dots, m$.
They form a complex basis of $\mathrm{Cl}(U)$.

Geodesic equation

Connection

Semi-Riemannian connection is defined by Koszul formula:

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(-\langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle\right),$$

X, Y, Z – arbitrary basis elements in $\mathcal{G}$.

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X, Y, Z – arbitrary basis elements in $\mathcal{G}$.

Geodesic

$\gamma\colon [0,1] \to G$ which satisfies the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$.

Geodesic equation

Define functions $t \mapsto v(t)$ and $t \mapsto u(t)$ such that
$\gamma(t) = \big(v(t), u(t)\big)$ and
$\dot{v}(t)$ and $\dot{u}(t)$ are projections of $\dot{\gamma}(t)$ onto V and U respectively.

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$$\gamma(t) = \big(v_1(t), \ldots, v_n(t), u_1(t), \ldots, u_m(t)\big)$$

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Initial conditions

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0,$$

$$v(0) = 0, \qquad u(0) = 0,$$

$$\dot{v}(0) = \dot{v}^0, \qquad \dot{u}(0) = \dot{u}^0.$$

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$$v(0) = 0, \qquad u(0) = 0,$$

$$\dot{v}(0) = \dot{v}^0, \qquad \dot{u}(0) = \dot{u}^0.$$

$$\Rightarrow \quad \begin{array}{l} \ddot{v} - \eta j(\dot{u}^0)\dot{v} = 0, \\ \dot{u} + \frac{1}{2}[\dot{v}, v] = \dot{u}^0. \end{array}$$

Geodesics

$\ddot{v} - \eta j(\dot{u}^0)\dot{v} = 0$ – sub-semi-Riemannian case,
$\ddot{v} - j(\dot{u}^0)\dot{v} = 0$ – sub-Riemannian case.

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Matrix j

$$j^t = -j, \quad j^2(\dot{u}^0) = -|\dot{u}^0|^2 I_V,$$

All eigenvalues of j are purely imaginary and come in pairs $\pm i|\dot{u}^0|$.

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$$\left(\begin{array}{ccc} \begin{array}{cc} 0 & |\dot{u}^0| \\ -|\dot{u}^0| & 0 \end{array} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{array}{cc} 0 & |\dot{u}^0| \\ -|\dot{u}^0| & 0 \end{array} \end{array}\right)$$

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Matrix $A = \eta j$

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If index p – odd:

2 real eigenvalues $\pm|\dot{u}^0|$,
$2r-2$ purely imaginary eigenvalues $\pm i|\dot{u}^0|$, r– odd,
$n-2r$ complex eigenvalues which come in quadriples $\pm(\alpha_k \pm i\beta_k)$, $\sqrt{\alpha_k^2+\beta_k^2} = |\dot{u}^0|$.

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$2r-2$ purely imaginary eigenvalues $\pm i|\dot{u}^0|$, r– odd,
$n-2r$ complex eigenvalues which come in quadriples $\pm(\alpha_k \pm i\beta_k)$, $\sqrt{\alpha_k^2+\beta_k^2} = |\dot{u}^0|$.

If index p – even:

No real eigenvalues,
$2r$ purely imaginary eigenvalues $\pm i|\dot{u}^0|$, r– even,
$n-2r$ complex eigenvalues which come in quadriples $\pm(\alpha_k \pm i\beta_k)$,
$\sqrt{\alpha_k^2+\beta_k^2} = |\dot{u}^0|$

Horizontal part of geodesic

$$v(t) = \left(v_{s_1}, v_{s_2}, v_{s+1_1}, v_{s+1_2}, \dots, v_{r_1}, v_{r_2}, v_{r+1_1}, v_{r+1_2}, v_{r+1_3}, v_{r+1_4},\right.$$
$$\left. \dots, v_{\frac{n}{4}+\frac{r}{2}_1}, v_{\frac{n}{4}+\frac{r}{2}_2}, v_{\frac{n}{4}+\frac{r}{2}_3}, v_{\frac{n}{4}+\frac{r}{2}_4}\right).$$

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$$\ldots, v_{\frac{n}{4}+\frac{r}{2}_1}, v_{\frac{n}{4}+\frac{r}{2}_2}, v_{\frac{n}{4}+\frac{r}{2}_3}, v_{\frac{n}{4}+\frac{r}{2}_4}\Big).$$

$$v_{s_1}(t) = \frac{1}{|\dot{u}^0|}\left(\dot{v}^0_{s_1} \sinh\left(|\dot{u}^0|t\right) + \dot{v}^0_{s_2} \cosh\left(|\dot{u}^0|t\right) - \dot{v}^0_{s_2}\right),$$
$$v_{s_2}(t) = \frac{1}{|\dot{u}^0|}\left(\dot{v}^0_{s_2} \sinh\left(|\dot{u}^0|t\right) + \dot{v}^0_{s_1} \cosh\left(|\dot{u}^0|t\right) - \dot{v}^0_{s_1}\right),$$

Horizontal part of geodesic

$$v_{j_1}(t) = \frac{1}{|\dot{u}^0|}\left(\dot{v}_{j_1}^0 \sin\left(|\dot{u}^0|t\right) - \dot{v}_{j_2}^0 \cos\left(|\dot{u}^0|t\right) + \dot{v}_{j_2}^0\right),$$

$$v_{j_2}(t) = \frac{1}{|\dot{u}^0|}\left(\dot{v}_{j_2}^0 \sin\left(|\dot{u}^0|t\right) + \dot{v}_{j_1}^0 \cos\left(|\dot{u}^0|t\right) - \dot{v}_{j_1}^0\right),$$

$$j = s+1, \dots, r.$$

Horizontal part of geodesic

$$\begin{gathered}
v_{j_1}(t) = \frac{1}{|\dot{u}^0|}\left(\dot{v}^0_{j_1}\sin\left(|\dot{u}^0|t\right) - \dot{v}^0_{j_2}\cos\left(|\dot{u}^0|t\right) + \dot{v}^0_{j_2}\right), \\
v_{j_2}(t) = \frac{1}{|\dot{u}^0|}\left(\dot{v}^0_{j_2}\sin\left(|\dot{u}^0|t\right) + \dot{v}^0_{j_1}\cos\left(|\dot{u}^0|t\right) - \dot{v}^0_{j_1}\right), \\
j = s+1, \dots, r.
\end{gathered}$$

$$\begin{gathered}
v_{k_1}(t) = A_k \sinh\alpha_k t \sin\beta_k t + B_k \cosh\alpha_k t \cos\beta_k t \\
+ B_k \sinh\alpha_k t \cos\beta_k t + A_k \cosh\alpha_k t \sin\beta_k t - B_k, \\
v_{k_2}(t) = -B_k \sinh\alpha_k t \sin\beta_k t + A_k \cosh\alpha_k t \cos\beta_k t \\
+ A_k \sinh\alpha_k t \cos\beta_k t - B_k \cosh\alpha_k t \sin\beta_k t - A_k, \\
A_k = \frac{1}{|\dot{u}^0|^2}(\alpha_k \dot{v}^0_{k_2} + \beta_k \dot{v}^0_{k_1}), \quad B_k = \frac{1}{|\dot{u}^0|^2}(\alpha_k \dot{v}^0_{k_1} - \beta_k \dot{v}^0_{k_2}),
\end{gathered}$$

Horizontal part of geodesic

$$\begin{aligned} v_{k_3}(t) &= -C_k \sinh \alpha_k t \sin \beta_k t - D_k \cosh \alpha_k t \cos \beta_k t \\ &+ D_k \sinh \alpha_k t \cos \beta_k t + C_k \cosh \alpha_k t \sin \beta_k t + D_k, \\ v_{k_4}(t) &= D_k \sinh \alpha_k t \sin \beta_k t - C_k \cosh \alpha_k t \cos \beta_k t \\ &+ C_k \sinh \alpha_k t \cos \beta_k t - D_k \cosh \alpha_k t \sin \beta_k t + C_k, \\ C_k &= \frac{1}{|\dot{u}^0|^2}(\alpha_k \dot{v}^0_{k_4} + \beta_k \dot{v}^0_{k_3}), \quad D_k = \frac{1}{|\dot{u}^0|^2}(\alpha_k \dot{v}^0_{k_3} - \beta_k \dot{v}^0_{k_4}). \end{aligned}$$

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A curve $v(t)$ has the same casual type as the initial velocity vector $\dot{v}^0$.

Horizontal part of geodesic

The projection of the curve $v(t)$ onto the (v_{s_1}, v_{s_2})-plane, is a brunch of the hyperbola with the canonical equation

$$\left(v_{s_1} + \frac{\dot{v}^0_{s_2}}{|\dot{u}^0|}\right)^2 - \left(v_{s_2} + \frac{\dot{v}^0_{s_1}}{|\dot{u}^0|}\right)^2 = \frac{-(\dot{v}^0_{s_1})^2 + (\dot{v}^0_{s_2})^2}{|\dot{u}^0|^2}.$$

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The projection of the curve $v(t)$ onto the (v_{j_1}, v_{j_2})-plane, $j = s+1, \dots, r$, is a circle with the center $\left(-\frac{\dot v^0_{j_2}}{|\dot u^0|}, \frac{\dot v^0_{j_1}}{|\dot u^0|}\right)$ of radius $\frac{\sqrt{(\dot v^0_{j_1})^2+(\dot v^0_{j_2})^2}}{|\dot u^0|}$.

Logarithmic spiral

The projection of the curve $v(t)$ onto the (v_{k_1}, v_{k_2})-plane, $k = 1, \dots, \frac{n}{4} + \frac{r}{2}$, is a logarithmic spiral with the equation

$$v_{k_1}^2(t) + v_{k_2}^2(t) = \frac{1}{|\dot{u}^0|^2}\Big((\dot{v}_{k_1}^0)^2 + (\dot{v}_{k_2}^0)^2\Big)e^{2\alpha_k t}.$$

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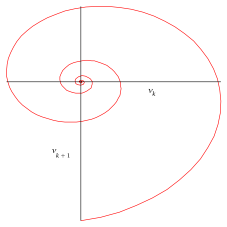

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