Stochastic Hamiltonian systems. Symmetries and skew-products.

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La Laguna, 02/2009

Part I

Stochastic Hamiltonian Systems

Motivations

Why stochastic Hamiltonian systems?

- Generalization of Bismut's "Mécanique Aléatoire" : Poisson manifolds and semimartingales as noise.
- Modeling of physical systems subjected to random perturbations.
- Modeling inaccuracy in the knowledge of physical parameters (random dynamical systems).
- Box-Jenkins: deterministic+stochastic for complex systems.
- Chaotic behavior of solutions and probabilistic treatment of dynamical evolution.
- Dissipation:

$$\text{microscopic collisions} \longrightarrow \text{macrosopic damping.}$$

Constraints:

- Respect geometry
- Variational principle

How do we do it?

- We replace the Hamiltonian differential equations by SDEs inspired by Emery's "transfer principle".
- Global formulation available:
  - Stratonovich integration of one-forms over semimartingales (Meyer)
  - Itô integration of second order one-forms over semimartingales (Schwartz)
- Use of the associated SDEs
- We will worry about the variational principle separately.

- Bismut: *Mécanique Aléatoire.* LNM 866. Springer.
- Lázaro-Camí, J.-A. and Ortega, J.-P. [2007] Stochastic Hamiltonian dynamical systems. http://arxiv.org/abs/math/0702787. To appear in *Rep. on Math. Phys.*
- Lázaro-Camí, J.-A. and Ortega, J.-P. [2007] Reduction, reconstruction, and skew-product decomposition of symmetric stochastic differential equations. http://arxiv.org/abs/0705.3156

Hamilton's equations in integral form

Setup: (M, ω) symplectic manifold or $(M, \{\cdot, \cdot\})$ Poisson manifold.

$B^\sharp : T^*M \to TM$ (respectively $\omega^\sharp : T^*M \to TM$) is the vector bundle map naturally associated to the Poisson tensor $B \in \Lambda^2(M)$ (respectively symplectic form $\omega \in \Omega^2(M)$) defined by

$$B(z)(\alpha_z, \beta_z) = \langle \alpha_z, B^\sharp(\beta_z) \rangle.$$

Proposition

(M, ω) a symplectic manifold and $h \in C^\infty(M)$. The smooth curve $\gamma : [0, T] \to M$ is an integral curve of the Hamiltonian vector field X_h if and only if for any $\alpha \in \Omega(M)$ and for any $t \in [0, T]$

$$\int_{\gamma|_{[0,t]}} \alpha = -\int_0^t \mathbf{d}h(\omega^\sharp(\alpha)) \circ \gamma(s) ds. \tag{1}$$

More generally, if M is a Poisson manifold with bracket $\{\cdot,\cdot\}$ then the same result holds with (1) replaced by

$$\int_{\gamma|_{[0,t]}} \alpha = -\int_0^t \mathbf{d}h(B^\sharp(\alpha)) \circ \gamma(s) ds, \qquad (2)$$

Reminder of the Stratonovich side

- M and N finite dimensional manifolds.
- $(\Omega, \mathcal{F}, \{\mathcal{F}_t \mid t \geq 0\}, P)$ filtered probability space.
- $X : \mathbb{R}_+ \times \Omega \to N$ a N-valued semimartingale.
- Stratonovich operator: family $\{S(x,y)\}_{x \in N, y \in M}$ such that $S(x,y) : T_x N \to T_y M$ is a linear mapping that depends smoothly on its two entries.
- $S^*(x,y) : T_y^* M \to T_x^* N$ the adjoint of $S(x,y)$.
- A M-valued semimartingale Γ solution of the the Stratonovich SDE

$$\delta\Gamma = S(X,\Gamma)\delta X$$

associated to X and S, if for any $\alpha \in \Omega(M)$:

$$\int \langle \alpha, \delta\Gamma \rangle = \int \langle S^*(X,\Gamma)\alpha, \delta X \rangle.$$

- We will refer to X as the noise semimartingale or the stochastic component of the stochastic differential equation.
- Existence. Uniqueness formulated through maximality of explosion times.

The Hamiltonian SDE

- $(M, \{\cdot,\cdot\})$ a Poisson manifold.
- $X : \mathbb{R}_+ \times \Omega \rightarrow V$ semimartingale with values on the vector space V with $X_0 = 0$.
- $h : M \rightarrow V^*$ a smooth function.
- $\{\epsilon^1, \ldots, \epsilon^r\}$ basis of V^* and $h = \sum_{i=1}^r h_i \epsilon^i$.
- The Hamilton equations with stochastic component X, and Hamiltonian function h are the Stratonovich stochastic differential equation

  $$\delta\Gamma^h = H(X,\Gamma)\delta X,$$

  defined by the Stratonovich operator $H(v,z) : T_v V \rightarrow T_z M$ given by

  $$H(v,z)(u) := \sum_{j=1}^r \langle \epsilon^j, u\rangle X_{h_j}(z).$$

Integral form and solutions

- $f : M \to W$. Define $\mathbf{d}f : TM \to W$ as $\mathbf{d}f = p_2 \circ Tf$, where $Tf : TM \to TW = W \times W$ is the tangent map of f and $p_2 : W \times W \to W$ is the projection onto the second factor. If $f = \sum_{i=1}^{n} f^i e_i$, then $\mathbf{d}f = \sum_{i=1}^{n} \mathbf{d}f^i \otimes e_i$.
- The dual Stratonovich operator $H^*(v,z) : T_z^*M \to T_v^*V$ of $H(v,z)$ is given by

$$H^*(v,z)(\alpha_z) = -\mathbf{d}h(z) \cdot B^\sharp(z)(\alpha_z).$$

Integral form and solutions

For any $\mathcal{F}_0$ measurable random variable Γ_0,there exists a unique semimartingale Γ^h such that $\Gamma^h_0 = \Gamma_0$ and a maximal stopping time ζ^h such that for any $\alpha \in \Omega(M)$,

$$\int \langle \alpha, \delta\Gamma^h \rangle = -\int \langle \mathbf{d}h(B^\sharp(\alpha))(\Gamma^h), \delta X \rangle.$$

We will refer to Γ^h as the **Hamiltonian semimartingale** associated to h with initial condition Γ_0.

In Darboux-Weinstein coordinates

$$
\begin{aligned}
q^i(\Gamma^h_\tau) - q^i(\Gamma^h_0) &= \sum_{j=1}^{r} \int_0^\tau \frac{\partial h_j}{\partial p_i}(\Gamma)\delta X^j, \\
p_i(\Gamma^h_\tau) - p_i(\Gamma^h_0) &= -\sum_{j=1}^{r} \int_0^\tau \frac{\partial h_j}{\partial q^i}(\Gamma)\delta X^j, \\
z_i(\Gamma^h_\tau) - z_i(\Gamma^h_0) &= \sum_{j=1}^{r} \int_0^\tau \{z_i, h_j\}_T(\Gamma)\delta X^j.
\end{aligned}
$$

Why Stratonovich?

- Respects geometry; Malliavin principle (only valid to a certain extent, as we will see with the variational principle)
- Mathematically more economical. The formulation of an Itô SDE via the Emery's transfer principle requires the choice of a connection to construct the Schwartz operator.

Time evolution of the observables

$f \in C^\infty(M)$ a function on phase space

- Stratonovich:

$$f(\Gamma^h_\tau) - f(\Gamma^h_0) = \sum_{j=1}^{r} \int_0^\tau \{f, h_j\}(\Gamma^h)\delta X^j$$

- Itô:

$$f\left(\Gamma^h_\tau\right) - f\left(\Gamma^h_0\right) = \sum_{j=1}^{r} \int_0^\tau \{f, h_j\}\left(\Gamma^h\right) dX^j + \frac{1}{2}\sum_{j,i=1}^{r} \int_0^\tau \{\{f, h_j\}, h_i\}\left(\Gamma^h\right) d\left[X^j, X^i\right]$$

First properties

- Energy is not automatically preserved. Resemblance with double bracket dissipation.
- Local preservation of symplectic leaves (up to a stopping time). Gobal preservation of their closure.
- Liouville's theorem.

Liouville's theorem

Let $\zeta : M \times \Omega \to [0, \infty]$ be the map such that, for any $z \in M$, $\zeta(z)$ is the maximal stopping time associated to the solution of the stochastic Hamilton equations with initial condition $\Gamma_0 = z$ a.s.. Let F be the **flow** such that for any $z \in M$,

$$F(z) : [0, \zeta(z)] \to M$$

is the solution semimartingale with initial condition z. The map $z \in M \longmapsto F_t(z, \omega) \in M$ is a local diffeomorphism of M, for each $t \geq 0$ and almost all $\omega \in \Omega$ in which this map is defined. Then, for any $z \in M$ and any $(t, \eta) \in [0, \zeta(z)]$,

$$F_t^*(z, \eta)\, \omega = \omega.$$

The importance of conservation laws

- They make easier the integration of the systems that have them. Sometimes associated to symmetries. Reduction.
- Provide qualitative information about the dynamics. Invariant manifolds.
- Help in concluding the nonlinear stability of certain equilibrium solutions using Dirichlet type criteria.

Definition

A function $f \in C^{\infty}(M)$ is said to be a **strongly** (respectively, **weakly**) **conserved quantity** of the stochastic Hamiltonian system associated to $h : M \to V^*$ if for any solution Γ^h of the stochastic Hamilton equations we have that $f\left(\Gamma^h\right) = f\left(\Gamma^h_0\right)$ (respectively, $E[f\left(\Gamma^h_\tau\right)] = E[f\left(\Gamma^h_0\right)]$, for any stopping time τ).

- Strongly conserved quantities are obviously weakly conserved.
- The two definitions coincide for deterministic systems with the standard definition of conserved quantity.

Conservation laws and Poisson involution

Proposition

$(M, \{\cdot,\cdot\})$ a Poisson manifold, $X : \mathbb{R}_+ \times \Omega \to V$ a semimartingale such that $X_0 = 0$, and $h : M \to V^*$ and $f \in C^\infty(M)$ two smooth functions. If $\{f, h_j\} = 0$ for every component h_j of h then f is a strongly conserved quantity of the stochastic Hamilton equations. Conversely, suppose that the semimartingale $X = \sum_{j=1}^r X^j \epsilon_j$ is such that $\left[X^i, X^j\right] = 0$ if $i \neq j$. If f is a strongly conserved quantity then $\{f, h_j\} = 0$, for any $j \in \{1, ..., r\}$ such that $\left[X^j, X^j\right]$ is an strictly increasing process at 0. The last condition means that there exists $A \in \mathcal{F}$ and $\delta > 0$ with $P(A) > 0$ such that for any $t < \delta$ and $\omega \in A$ we have $[X^j, X^j]_t(\omega) > [X^j, X^j]_0(\omega)$, for all $j \in \{1, ..., r\}$.

Lyapunov stability

Definition

Given $x \in M$ and $s \in \mathbb{R}$, denote by $\Gamma^{s,x}$ the unique solution of a given SDE such that $\Gamma^{s,x}_s(\omega) = x$, for all $\omega \in \Omega$. A point $z_0 \in M$ is an **equilibrium** of the SDE if the constant process $\Gamma_t(\omega) := z_0$ is a solution, for all $t \in \mathbb{R}$ and $\omega \in \Omega$. We say that the equilibrium z_0 is

(i) **Almost surely (Lyapunov) stable** when for any open neighborhood U of z_0 there exists another neighborhood $V \subset U$ of z_0 such that for any $z \in V$ we have $\Gamma^{0,z} \subset U$, a.s.

(ii) **Stable in probability.** For any $s \geq 0$ and $\epsilon > 0$

$$\lim_{x \longrightarrow z_0} P \left\{ \sup_{t>s} d\left(\Gamma^{s,x}_t, z_0\right) > \epsilon \right\} = 0,$$

Theorem

In the setup of the previous definition, assume that there exists a function $f \in C^\infty(M)$ such that $\mathbf{d}f(z_0) = 0$ and that the quadratic form $\mathbf{d}^2 f(z_0)$ is (positive or negative) definite. If f is a strongly (respectively, weakly) conserved quantity then the equilibrium z_0 is almost surely stable (respectively, stable in probability).

Lyapunov functions

Definition

Let U be an open neighborhood of the equilibrium z_0 and let $V : U \longrightarrow \mathbb{R}$ be a continuous function. V is a **Lyapunov function** for the equilibrium z_0 if $V(z_0) = 0$, $V(z) > 0$ for any $z \in U \setminus \{z_0\}$, and

$$E[V(\Gamma_\tau)] \leq E[V(\Gamma_0)], \tag{3}$$

for any stopping time τ and any solution Γ of the SDE.

This definition generalizes to the stochastic context the standard notion of Lyapunov function that one encounters in dynamical systems theory. If the stochastic differential equation in question is associated to an Itô diffusion and the Lyapunov function is twice differentiable, the inequality (3) can be ensured by requiring that $A[V](z) \leq 0$, for any $z \in U \setminus \{z_0\}$, where A is the infinitesimal generator of the diffusion, and by using Dynkin's formula.

Stochastic Lyapunov's Theorem

Theorem

Let $z_0 \in M$ be an equilibrium solution of the stochastic differential equation and let $V : U \longrightarrow \mathbb{R}$ be a continuous Lyapunov function for z_0. Then z_0 is stable in probability.

Stochastic perturbation of an integrable system

Consider the system in action-angle variables $(I_1, \ldots, I_n, \theta_1, \ldots, \theta_n)$

$$d\Gamma = X_{h_0} \circ dt + \epsilon \sum_{i=1}^{n} X_{h_i} \circ dB^i, \quad \epsilon > 0$$

- $h_0(\mathbf{I}) = \langle \mathbf{I}, A\mathbf{I} \rangle$, with A a regular $n \times n$ matrix.
- $X_{h_i} = \frac{\partial}{\partial I_i}$, locally Hamiltonian vector field.
- Since $\frac{\partial^2 h_0}{\partial I_i \partial I_j} = A_{ij}$ and the frequency map $A : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism ($\det A \neq 0$), KAM theorem applies to Hamiltonian deterministic perturbations of X_{h_0}.
- We study the solution semimartingale Γ with the support theorems and Hörmander's condition.

The support theorems and Hörmander's condition

The support theorems (Stroock-Varadhan-Kunita...) study the support of the law of Γ in terms of the controllability of the system of ordinary controlled differential equations

$$\frac{dx(t)}{dt} = X_{h_0}(x(t)) + \epsilon \sum_{i=1}^{n} X_{h_i}(x(t)) u^i(t).$$

- Complete controllability translates into having a non-zero probability of finding the system in any given region at some time in the future (classical support theorem).
- Strong complete controllability translates into having a non-zero probability of finding the system in any given region at any GIVEN time in the future (Kunita).

Total destruction of tori

Strong complete controllability implies the total destruction of invariant tori at arbitrarily small time $t > 0$. And that is what we have!!!

- Consider three Lie algebras $\mathcal{B} \subset \mathcal{I} \subset \mathcal{L}$.
- $\mathcal{B} = \mathrm{Lie}\{X_{h_1}, \ldots, X_{h_n}\}$.
- $\mathcal{L} = \mathrm{Lie}\{X_{h_0}, X_{h_1}, \ldots, X_{h_n}\}$.
- $\mathcal{I}$ is the ideal in $\mathcal{L}$ generated by $\{X_{h_1}, \ldots, X_{h_n}\}$.

Kunita's Theorem

We have strong complete controllability provided that:

(i) $\dim \mathcal{I}(x) = 2n$, for any x (Hörmander's condition).

(ii) $[\mathcal{B}, \mathcal{I}](x) \subset \mathcal{B}(x)$, for any x.

(iii) $\mathcal{B}$ is locally finitely generated.

In our case

Note that $[X_{h_i}, X_{h_j}] = 0$, $i, j \in \{1, \ldots, n\}$, $[X_{h_0}, X_{h_i}] = \sum_{j=1}^{n} A_{ij} \frac{\partial}{\partial \theta_j}$, and $[[X_{h_0}, X_{h_i}], X_{h_j}] = 0$, hence

- $\mathcal{B} = \mathrm{span}\{X_{h_1}, \ldots, X_{h_n}\}$
- $\mathcal{I} = \mathrm{span}\{X_{h_1}, \ldots, X_{h_n}, [X_{h_0}, X_{h_1}], \ldots, [X_{h_0}, X_{h_n}]\}$
- $\mathcal{L} = \mathrm{span}\{X_{h_0}, X_{h_1}, \ldots, X_{h_n}, [X_{h_0}, X_{h_1}], \ldots, [X_{h_0}, X_{h_n}]\}$

Since A is invertible it is clear that $\mathcal{I}(x) = T_x M$, $\mathcal{B}$ is obviously finitely generated, and as $[\mathcal{B}, \mathcal{I}] = 0$, point (ii) is also satisfied.

Bismut's Hamiltonian diffusions

- $(M, \{\cdot, \cdot\})$ a Poisson manifold.
- $h_j \in C^\infty(M)$, $j = 0, ..., r$, smooth functions.
- $h : M \longrightarrow \mathbb{R}^{r+1}$, the Hamiltonian function $m \longmapsto (h_0(m), \ldots, h_r(m))$.
- Semimartingale $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{r+1}$ given by:

$$(t, \omega) \longmapsto \left(t, B_t^1(\omega), \ldots, B_t^r(\omega)\right),$$

  where B^j, $j = 1, ..., r$, are r-independent Brownian motions.
- In this setup, for any $f \in C^\infty(M)$

$$f\left(\Gamma_\tau^h\right) - f\left(\Gamma_0^h\right) = \int_0^\tau \{f, h_0\}\left(\Gamma^h\right) dt + \sum_{j=1}^r \int_0^\tau \{f, h_j\}\left(\Gamma^h\right) dB^j + \int_0^\tau \{\{f, h_j\}, h_j\}\left(\Gamma^h\right) dt.$$

This equation may be interpreted as a stochastic perturbation of the classical Hamilton equations associated to h_0, that is,

$$\frac{d(f \circ \gamma)}{dt}(t) = \{f, h_0\}(\gamma(t)).$$

by the r Brownian motions B^j. These equations have been studied by Bismut when the Poisson manifold $(M, \{\cdot,\cdot\})$ is just the symplectic Euclidean space $\mathbb{R}^{2n}$ with the canonical symplectic form. He refers to these processes as Hamiltonian diffusions.

Notice that the classical energy does not need to be conserved. Taking averages:

$$\left.\frac{d}{dt}\right|_{t=s} E[h_0(\Gamma_t^h)] = \frac{1}{2}\sum_{i=1}^{r} E[\{\{h_0, h_i\}, h_i\}(\Gamma_s^h)].$$

Langevin equation and damping

$$m\, d\dot{q}(t) = -\lambda \dot{q}(t) dt + b dB_t$$

$\lambda > 0$ damping coefficient and B_t is a Brownian motion.

- Physical interpretation: the Brownian motion models random instantaneous bursts of momentum that are added to the particle by collision with lighter particles, while the mean effect of the collisions is the slowing down of the particle. This fact is mathematically described by noting that the expected value $q_e := E[q]$ satisfies the ODE $\ddot{q}_e = -\lambda \dot{q}_e$.
- This description is accurate but does not provide any information about the mechanism that links the presence of the Brownian perturbation to the emergence of damping:

$$\lambda \text{ should be } 0 \text{ when } b = 0!!!$$

The motion of a particle of mass m in one dimension subjected to viscous damping and to a harmonic potential with Hooke constant k satisfies the same differential equation as the expected value of the solution semimartingale of a natural stochastic Hamiltonian system, which provides a mathematical mechanism by which the stochastic perturbations in the system generate an average damping.

Consider the stochastic Hamiltonian system:

- $\mathbb{R}^2$ with its canonical symplectic two-form.
- $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ given by $X_t(\omega) := t + \nu B_t(\omega)$, where $\nu \in \mathbb{R}$ and B_t a Brownian motion.
- $h \in C^\infty(M)$ is the energy of the harmonic oscillator

$$h = \frac{1}{2m}p^2 + \frac{1}{2}kq^2.$$

- Writing $q_e := E[q(\Gamma_t^h)]$ it holds that

$$m\ddot{q}_e(t) = -\nu^2 k \dot{q}_e(t) - K\left(\frac{\nu^4 k}{4m} + 1\right) q_e(t).$$

- If $\nu = 0$ the damping vanishes and we get a free harmonic oscillator.

Integrable stochastic Hamiltonian systems

(M,ω), $2n$-dimensional manifold, $X:\mathbb{R}_+\times\Omega\to V$ a semimartingale, and $h:M\to V^*$ such that $h=\sum_{i=1}^r h_i\epsilon^i$, with $\left\{\epsilon^1,...,\epsilon^r\right\}$ a basis of V^*. H is the associated Stratonovich operator.

Suppose that there exists a family of functions $\{f_{r+1},\ldots,f_n\}\subset C^\infty(M)$ such that the n-functions $\{f_1:=h_1,...,f_r:=h_r,f_{r+1},...,f_n\}\subset C^\infty(M)$ are in Poisson involution, that is, $\{f_i,f_j\}=0$, for any $i,j\in\{1,...,n\}$. Assume also that $F:=(f_1,...,f_n)$ satisfies the hypotheses of the Liouville-Arnold Theorem: F has compact and connected fibers and its components are independent. In this setup, we say that the stochastic Hamiltonian dynamical system associated to H is **integrable**.

As it is the case for standard (Liouville-Arnold) integrable systems, there is a symplectomorphism that takes (M, ω) to $\left(\mathbb{T}^n \times \mathbb{R}^n, \sum_{i=1}^n \mathbf{d}\theta^i \wedge \mathbf{d}I_i\right)$ and for which $F \equiv F(I_1, ..., I_n)$. In particular, in the action-angle coordinates $\left(I_1, ..., I_n, \theta^1, ..., \theta^n\right)$, $h_j \equiv h_j(I_1, ..., I_n)$ with $j \in \{1, ..., r\}$. For any random variable Γ_0 and any $i \in \{1, ..., n\}$

$$
\begin{aligned}
I_i(\Gamma) - I_i(\Gamma_0) &= \sum_{j=1}^{r} \int \{I_i, h_j(\mathbf{I})\}(\Gamma)\, \delta X^j = 0 \\
\theta^i(\Gamma) - \theta^i(\Gamma_0) &= \sum_{j=1}^{r} \int \left\{\theta^i, h_j(\mathbf{I})\right\}(\Gamma)\, \delta X^j = \sum_{j=1}^{r} \int \frac{\partial h_j}{\partial I_i}(\Gamma)\, \delta X^j.
\end{aligned}
$$

The tori $\mathbf{I}=$ constant are left invariant by the stochastic flow. In particular, as the paths of the solutions are contained in compact sets, the stochastic flow is defined for any time and the flow is complete. Moreover, the restriction of this stochastic differential equation to the torus given by say, $\mathbf{I}_0$, yields the solution

$$\theta^i\left(\Gamma\right)-\theta^i\left(\Gamma_0\right)=\sum_{j=1}^{r}\omega_j\left(\mathbf{I}_0\right)X^j, \qquad (4)$$

where $\omega_j\left(\mathbf{I}_0\right):=\frac{\partial h_j}{\partial I_i}\left(\mathbf{I}_0\right)$ $(X_0=0)$.

The Haar measure $\mathbf{d}\theta^1\wedge\ldots\wedge\mathbf{d}\theta^n$ on each invariant torus is left invariant by the stochastic flow. If we can ensure that there exists a unique invariant measure μ (for instance, if (4) defines a non-degenerate diffusion on $\mathbb{T}^n$ (see Ikeda-Watanabe)) then μ coincides necessarily with the Haar measure.

Brownian motion on manifolds as Hamiltonian semimartingales

A M-valued process Γ is a **Brownian motion** on (M, g), with g a Riemannian metric M when Γ is continuous and adapted and for every $f \in C^{\infty}(M)$

$$f(\Gamma) - f(\Gamma_0) - \frac{1}{2} \int \Delta_M f(\Gamma) dt$$

is a local martingale.
Recall that the **Laplacian** $\Delta_M(f)$ is defined as

$$\Delta_M(f) = \mathrm{Tr}(\mathrm{Hess}\, f),$$

for any $f \in C^{\infty}(M)$, where $\mathrm{Hess}\, f := \nabla(\nabla f)$, with

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

the Levi-Civita connection of g.

The parallelizable case

Suppose that (M, g) is parallelizable (like any Lie group) and let $\{Y_1, ..., Y_n\}$ be an orthonormal parallelization. We construct a stochastic Hamiltonian system on the cotangent bundle T^*M of M, endowed with its canonical symplectic structure, such that the projection of the solution semimartingales of this system onto M are M-valued Brownian motions.

- Noise semimartingale: $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n+1}$ given by $X(t,\omega) := (t, B_t^1(\omega), \ldots, B_t^n(\omega))$, with B^j, $j = 1, ..., n$, are n-independent Brownian motions.
- Hamiltonian function $h = (h_0, h_1, \ldots, h_n) : T^*M \to \mathbb{R}^{n+1}$ the function whose components are given by

$$h_0 : \begin{array}{ccc} T^*M & \longrightarrow & \mathbb{R} \\ \alpha_m & \longmapsto & -\frac{1}{2}\sum_{j=1}^n \langle \alpha_m, \left(\nabla_{Y_j} Y_j\right)(m)\rangle, \end{array} \qquad h_j : \begin{array}{ccc} T^*M & \longrightarrow & \mathbb{R} \\ \alpha_m & \longmapsto & \langle \alpha_m, Y_j(m)\rangle. \end{array}$$

It can be checked that for any function of the type $\overline{f} \circ \pi$ with $\pi : T^*M \to M$ and $\overline{f} \in C^\infty(M)$

$$\overline{f} \circ \pi \left(\Gamma^h\right) - \overline{f} \circ \pi \left(\Gamma_0^h\right) - \frac{1}{2} \int \Delta_M(\overline{f}) \left(\pi \circ \Gamma^h\right) dt = \sum_{j=1}^{n} \int g \left(\operatorname{grad} \overline{f}, Y_j\right) \left(\pi \circ \Gamma^h\right) dB_s^j. \quad (5)$$

Since $\sum_{i=1}^n \int g \left(\operatorname{grad} \overline{f}, Y_j\right) (\overline{\Gamma}^h) dB^i$ is a local martingale, $\pi(\Gamma^h)$ is a Brownian motion.

Brownian motions on Lie groups

G a Lie group with Lie algebra $\mathfrak{g}$ and assume that G admits a bi-invariant metric g (for example when G is Abelian or compact). This metric induces a pairing in $\mathfrak{g}$ invariant with respect to the adjoint representation of G on $\mathfrak{g}$. Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to this invariant pairing and let $\{\nu_1, \ldots, \nu_n\}$ be the corresponding dual basis of $\mathfrak{g}^*$. The infinitesimal generator vector fields $\{\xi_{1G}, \ldots, \xi_{nG}\}$

$$\xi_{iG}(h) = T_e L_h \cdot \xi,$$

are an orthonormal parallelization of G, that is $g(\xi_{iG}, \xi_{jG}) := \delta_{ij}$. Since g is bi-invariant then

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad X, Y \in \mathfrak{X}(G)$$

hence $\nabla_{\xi_{iG}}\xi_{iG} = 0$ and $h_0 \equiv 0$ we can therefore take $h_G = (h_1, ..., h_n)$ and $X_G = \left(B_t^1, ..., B_t^n\right)$ when we consider the Hamilton equations of the Brownian motion with respect to g.

Brownian motion on a circle: $S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. The stochastic Hamiltonian differential equation for the semimartingale Γ^h associated to $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, given by $X_t(\omega) := B_t(\omega)$, and the Hamiltonian function $h : TS^1 \simeq S^1 \times \mathbb{R} \to \mathbb{R}$ given by $h(e^{i\theta}, \lambda) := \lambda$, is simply obtained by writing (5) down for the functions $f_1(e^{i\theta}) := \cos\theta$ and $f_2(e^{i\theta}) := \sin\theta$ which provide us with the equations for the projections X^h and Y^h of Γ^h onto the OX and OY axes, respectively. A straightforward computation yields

$$dX^h = -Y^h dB - \frac{1}{2}X^h dt \quad \text{and} \quad dY^h = X^h dB - \frac{1}{2}Y^h dt.$$

A solution is $(X_t^h, Y_t^h) = (\cos B_t, \sin B_t)$, that is, $\Gamma_t^h = e^{iB_t}$.

Brownian motions on arbitrary manifolds

Let (M, g) be a not necessarily parallelizable Riemannian manifold. We will replace the cotangent bundle of the manifold by the cotangent bundle of its orthonormal frame bundle

$$\mathcal{O}(M) = \bigcup_{x \in M} \mathcal{O}_x(M)$$

which is a smooth manifold of dimension $n(n+1)/2$ such that $\mathcal{O}_x(M)$ is the set of orthonormal frames for T_xM.
A curve $\gamma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to \mathcal{O}(M)$ is called horizontal if γ_t is the parallel transport of γ_0 along the projection $\pi(\gamma_t)$. The set of tangent vectors of horizontal curves that contain a point $u \in \mathcal{O}(M)$ defines the horizontal subspace $H_u\mathcal{O}(M) \subset T_u\mathcal{O}(M)$, with dimension n. The projection $\pi : \mathcal{O}(M) \to M$ induces an isomorphism $T_u\pi : H_u\mathcal{O}(M) \to T_{\pi(u)}M$.

On the orthonormal frame bundle, we have n horizontal vector fields Y_i, $i = 1, ..., n$, defined as follows: for each $u \in \mathcal{O}(M)$, let $Y_i(u)$ be the unique horizontal vector in $H_u\mathcal{O}(M)$ with $T_u\pi(Y_i) = u_i$, where u_i is the ith unit vector of the orthonormal frame u.
Introduce the functions $h_i : T^*\mathcal{O}(M) \to \mathbb{R}$, given by $h_i(\alpha) = \langle \alpha, Y_i \rangle$. Recall that $T^*\mathcal{O}(M)$ being a cotangent bundle it has a canonical symplectic structure. It can be seen that the Hamiltonian semimartingale Γ^h associated to $h = (h_1, ..., h_n)$ and to the Hamiltonian equations on $T^*\mathcal{O}(M)$ with stochastic component $X = \left(B_t^1, ..., B_t^n\right)$ is such that $U^h = \pi_{T^*\mathcal{O}(M)}\left(\Gamma^h\right)$ is a solution of the stochastic differential equation given Eells-Elworthy-Malliavin to characterize the Brownian motion. Hence, $X^h = \pi\left(U^h\right)$ is a Brownian motion on M.

Geometric Brownian motion

$B_1, \ldots, B_n$ be n-independent Brownian motions. The n-dimensional geometric Brownian motion is the SDE

$$dq_i = \mu_i q_i dt + q_i \sum_{j=1}^{n} \sigma_{ij} dB_j, \quad i = 1, \ldots, n. \tag{6}$$

Of much importance in mathematical finance: it models the behavior of n-stocks in an arbitrage-free and complete market in the context of the Black and Scholes formula.

$$h(\mathbf{q},\mathbf{p}) = \begin{pmatrix} \alpha_1 q_1 p_1 & \sigma_{11} q_1 p_1 & \cdots & \sigma_{1n} q_1 p_1 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n q_n p_n & \sigma_{n1} q_n p_n & \cdots & \sigma_{nn} q_n p_n \end{pmatrix} \text{ and } X_t = \begin{pmatrix} t & B_1 & \cdots & B_n \\ \vdots & \vdots & \ddots & \vdots \\ t & B_1 & \cdots & B_n \end{pmatrix},$$

where $\alpha_i = \mu_i - \frac{1}{2}\left(\sigma_{i1}^2 + \cdots + \sigma_{in}^2\right)$, $i = 1, \ldots, n$.

The stochastic action

The stochastic Hamilton equations satisfy a variational principle that generalizes the classical deterministic one. We shall consider an exact symplectic manifold (M, ω), that is, $\omega = -\mathbf{d}\Theta$ (the cotangent bundle T^*Q of any manifold Q, with Θ the Liouville one-form).

- $\mathcal{S}(M)$ and $\mathcal{S}(\mathbb{R})$ the M and real-valued semimartingales.
- **Stochastic action** associated to h: $S : \mathcal{S}(M) \to \mathcal{S}(\mathbb{R})$ given by

$$S(\Gamma) = \int \langle \Theta, \delta\Gamma \rangle - \int \left\langle \widehat{h}(\Gamma), \delta X \right\rangle,$$

where $\widehat{h}(\Gamma) : \mathbb{R}_+ \times \Omega \to V \times V^*$ is

$$\widehat{h}(\Gamma)(t, \omega) := (X_t(\omega), h(\Gamma_t(\omega))).$$

Directional derivative

Let $F : \mathcal{S}(M) \to \mathcal{S}(\mathbb{R})$ a map, and $\Gamma \in \mathcal{S}(M)$. We say that F is **differentiable** at Γ in the direction of a local one parameter group of diffeomorphisms $\varphi_s : (-\varepsilon, \varepsilon) \times M \to M$, if for any sequence $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, such that $s_n \underset{n\to\infty}{\longrightarrow} 0$, the family

$$X_n = \frac{1}{s_n}\left(F\left(\varphi_{s_n}(\Gamma)\right) - F(\Gamma)\right)$$

converges uniformly on compacts in probability (ucp) to a process that we will denote by $\left.\frac{d}{ds}\right|_{s=0} F\left(\varphi_s(\Gamma)\right)$ and that is referred to as the **directional derivative** of F at Γ in the direction of φ_s.

Characterization of the directional derivative

$F : \mathcal{S}(M) \to \mathcal{S}(\mathbb{R})$ the map defined by $F(\Gamma) := \int \langle \alpha, \delta\Gamma \rangle$. Then F is differentiable in all directions and

$$\left.\frac{d}{ds}\right|_{s=0} F(\varphi_s(\Gamma)) = \int \langle L_Y \alpha, \delta\Gamma \rangle .$$

For the action:

$$\left.\frac{d}{ds}\right|_{s=0} S(\varphi_s(\Gamma))$$
$$= -\int \langle \alpha, \delta\Gamma \rangle - \int \left\langle \mathbf{d}h\left(\omega^{\#}(\alpha)\right)(\Gamma), \delta X \right\rangle + \mathbf{i}_Y \Theta(\Gamma) - \mathbf{i}_Y \Theta(\Gamma_0).$$

Noether's theorem If the action is invariant by φ_s, that is, $S(\varphi_s(\Gamma)) = S(\Gamma)$, then the function $\mathbf{i}_Y \Theta$ is a conserved quantity of the stochastic Hamiltonian system associated to $h : M \to V^*$.

Adapted vector fields and variations

M a manifold and D a set. A local one parameter group of diffeomorphisms $\varphi : (-\varepsilon, \varepsilon) \times M \to M$ is **adapted** to D if $\varphi_s(y) = y$ for any $y \in D$ and any $s \in (-\varepsilon, \varepsilon)$. The corresponding vector field satisfies that $Y|_D = 0$ and is also called adapted to D. Let $\Gamma : \mathbb{R}_+ \times \Omega \to M$ be a M-valued continuous and adapted stochastic process. We will denote by

$$\tau_D = \inf \{ t > 0 \mid \Gamma_t(\omega) \notin D \}$$

the **first exit time** of Γ with respect to D. τ_D is a stopping time if D is measurable.

Critical Action Principle (Weak version)

$(M, \omega = -d\Theta)$ exact symplectic manifold, $X : \mathbb{R}_+ \times \Omega \to V$ a semimartingale such that $X_0 = 0$, and $h : M \to V^*$ a Hamiltonian function. Let $m_0 \in M$ be a point in M and $\Gamma : \mathbb{R}_+ \times \Omega \to M$ a continuous semimartingale defined on $[0, \zeta_\Gamma)$ such that $\Gamma_0 = m_0$. Suppose that there exist a measurable set U containing m_0 such that $\tau_U < \zeta_\Gamma$ a.s.. If the semimartingale Γ satisfies the stochastic Hamilton equations (with initial condition $\Gamma_0 = m_0$) on the interval $[0, \tau_U]$ then for any local one parameter group of diffeomorphisms $\varphi : (-\varepsilon, \varepsilon) \times M \to M$ adapted to $\{m_0\} \cup \partial U$ we have

$$\mathbf{1}_{\{\tau_U < \infty\}} \left[\left.\frac{d}{ds}\right|_{s=0} S\left(\varphi_s\left(\Gamma\right)\right)\right]_{\tau_U} = 0 \quad \text{a.s..}$$

Important: This critical action principle does not admit a converse. We need more general variations.

Pathwise variations

Γ a M-valued semimartingale. Let $s_0 > 0$; we say that the map $\Sigma : (-s_0, s_0) \times \mathbb{R}_+ \times \Omega \to M$ is a **pathwise variation** of Γ whenever $\Sigma_t^0 = \Gamma_t$ for any $t \in \mathbb{R}_+$ a.s.. The pathwise variation Σ of Γ **converges uniformly** to Γ whenever:

(i) For any $f \in C^\infty(M)$, $f(\Sigma^s) \to f(\Gamma)$ in *ucp* as $s \to 0$.

(ii) There exists a process $Y : \mathbb{R}_+ \times \Omega \to TM$ over Γ such that, for any $f \in C^\infty(M)$, the Stratonovich integral $\int Y[f]\,\delta X$ exists for any X (for instance if Y is a semimartingale) and the increments $(f(\Sigma^s) - f(\Gamma))/s$ converge in *ucp* to $Y[f]$ as $s \to 0$. We will call such a Y the **infinitesimal generator** of Σ.

Critical Action Principle

$(M, \omega = -d\theta)$ exact symplectic manifold, $X : \mathbb{R}_+ \times \Omega \to V$, and $h : M \to V^*$ a Hamiltonian function. Let m_0 be a point in M and $\Gamma : \mathbb{R}_+ \times \Omega \to M$ a continuous adapted semimartingale defined on $[0, \zeta_\Gamma)$ such that $\Gamma_0 = m_0$. $K \subseteq M$ a compact set that contains m_0 and τ_K the first exit time of Γ from K. Suppose that $\tau_K < \infty$ a.s..

(i) For any bounded pathwise variation Σ with bounded infinitesimal generator Y which converges uniformly to Γ^{τ_K} uniformly, the action has a directional derivative that equals

$$\left.\frac{d}{ds}\right|_{s=0} S(\Sigma^s) := \int \langle \mathbf{i}_Y d\theta, \delta\Gamma^{\tau_K}\rangle - \int \left\langle \widehat{Y[h]}(\Gamma^{\tau_K}), \delta X \right\rangle + \langle \theta(\Gamma^{\tau_K}), Y\rangle - \langle \theta(\Gamma^{\tau_K}), Y\rangle_{t=0}$$

(ii) The semimartingale Γ satisfies the stochastic Hamiltonian equations with initial condition $\Gamma_0 = m_0$ up to time τ_K if and only if, for any bounded pathwise variation $\Sigma : (-s_0, s_0) \times \mathbb{R}_+ \times \Omega \longrightarrow M$ with bounded infinitesimal generator which converges uniformly to Γ^{τ_K} and such that $\Sigma_0^s = m_0$ and $\Sigma_{\tau_K}^s = \Gamma_{\tau_K}$ a.s. for any $s \in (-s_0, s_0)$,

$$\left[\left. \frac{d}{ds} \right|_{s=0} S\left(\Sigma^s\right) \right]_{\tau_K} = 0 \quad \text{a.s.}.$$

Part II

Symmetries, reduction, and reconstruction in SDEs

Symmetries and degeneracies

Let $X : \mathbb{R}_+ \times \Omega \longrightarrow N$ be a N-valued semimartingale and let $S : TN \times M \longrightarrow TM$ be a Stratonovich operator. Let $\phi : M \longrightarrow M$ be a diffeomorphism. We say that ϕ is a **symmetry** of the stochastic differential equation

$$\delta\Gamma = S(X, \Gamma)\delta X$$

if for any $x \in N$ and $y \in M$

$$S\left(x, \phi\left(y\right)\right) = T_y\phi \circ S\left(x, y\right).$$

As in the standard deterministic context, the symmetries of a stochastic differential equation imply degeneracies at the level of its solutions:

Proposition

Let $X : \mathbb{R}_+ \times \Omega \longrightarrow N$ be a N-valued semimartingale, $S : TN \times M \longrightarrow TM$ a Stratonovich operator, and let $\phi : M \longrightarrow M$ be a symmetry of the corresponding stochastic differential equation. If Γ is solution then so is $\phi(\Gamma)$.

The symmetries that we are mostly interested in are induced by the action of a Lie group G on the manifold M via the map $\Phi : G \times M \longrightarrow M$. We will usually write $g \cdot z := \Phi(g, z)$. Also

$$\begin{array}{rccc} \Phi_z : G & \longrightarrow & M \\ g & \longmapsto & g \cdot z \end{array}, \qquad \begin{array}{rccc} \Phi_g : M & \longrightarrow & M \\ z & \longmapsto & g \cdot z \end{array}.$$

$\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g} \cdot m := T_m(G \cdot m)$.

Definition

We will say that the stochastic differential equationis G-**invariant** if, for any $g \in G$, the diffeomorphism $\Phi_g : M \rightarrow M$ is a symmetry in the sense of the previous definition.

Given a solution Γ of a G-invariant stochastic differential equation, the previous proposition provides an entire orbit of solutions since for any $g \in G$, the semimartingale $\Phi_g(\Gamma)$ is also a solution.

Degeneracies of the probability law

$\Gamma : \{0 \leq t < \zeta\} \to M$ solution of the G-invariant system (M, S, X, N) defined up to the explosion time ζ (maybe finite if M is not compact). In such case consider Γ as a process that takes values in the Alexandroff one-point compactification $\hat{M} := M \cup \{\infty\}$ of M and it is hence defined in the whole space $\mathbb{R}_+ \times \Omega$. In this picture, the process Γ is continuous and such that $\Gamma_t(\omega) = \{\infty\}$, for any $(t, \omega) \in \mathbb{R}_+ \times \Omega$ such that $t \geq \zeta(\omega)$. Let $\hat{W}(M)$ be the path space defined by

$$\begin{aligned}\hat{W}(M) = \{w : [0,\infty] \to \hat{M} \text{ continuous such that } w(0) \in M \text{ and}\\ \text{if } w(t) = \{\infty\} \text{ then } w(t') = \{\infty\} \text{ for any } t' \geq t\}.\end{aligned}$$

Let $\{P_z \mid z \in M\}$ be the family of probability measures on $\hat{W}(M)$ defined by the solutions of (M, S, X, N), that is, P_z is the law of the random variable $\Gamma^z : \Omega \longrightarrow \hat{W}(M)$, where Γ^z is the solution of (M, S, X, N) with initial condition $\Gamma^z_{t=0} = z$ a.s.. The action $\Phi : G \times M \longrightarrow M$ may be extended to $\hat{M}$ setting $\Phi_g(\{\infty\}) = \{\infty\}$ for any $g \in G$. It is easy to check that

$$P_{g \cdot z} = \Phi_g^* P_z,$$

that is, for any measurable set $A \subset \hat{W}(M)$, $P_{g \cdot z}(A) = P_z(\Phi_g(A))$.

Invariant submanifolds

The presence of symmetry in a stochastic differential equation is also associated with the occurrence of conserved quantities and, more generally, with the appearance of invariant submanifolds.

Definition

Γ a solution of the stochastic differential equation $\delta\Gamma = S(X,\Gamma)\delta X$ and L an injectively immersed submanifold of M. Let ζ be the maximal stopping time of Γ and suppose that $\Gamma_0(\omega) = Z_0$, where Z_0 is a random variable such that $Z_0(\omega) \in L$, for all $\omega \in \Omega$. We say that L is an **invariant submanifold** of the stochastic differential equation if for any stopping time $\tau < \zeta$ we have that $\Gamma_\tau \in L$.

Proposition

Let $X : \mathbb{R}_+ \times \Omega \to N$ be a N-valued semimartingale and let $S : TN \times M \to TM$ be a Stratonovich operator. Let L be an injectively immersed submanifold of M and suppose that the Stratonovich operator S is such that

$$\mathrm{Im}\,(S(x,y)) \subset T_y L, \quad \text{for any } y \in L \text{ and any } x \in N.$$

Then, the closure of L is an invariant subset of the associated stochastic differential equation.

Corollary

Law of conservation of the isotropy $X : \mathbb{R}_+ \times \Omega \to N$ semimartingale and $S : TN \times M \to TM$ a Stratonovich operator that is invariant with respect to a proper action of the Lie group G on M. Then, for any isotropy subgroup $I \subset G$, the isotropy type submanifolds $M_I := \{z \in M \mid G_z = I\}$ are locally invariant submanifolds of the associated stochastic differential equation.

Symmetry reduction

Theorem

$X : \mathbb{R}_+ \times \Omega \to N$ semimartingale and $S : TN \times M \to TM$ a G-invariant Stratonovich operator (proper action). $I \subset G$ an isotropy subgroup, M_I the isotropy type submanifold, and $L_I := N(I)/I$. The orbit space M_I/L_I is a regular quotient manifold. There is a well defined Stratonovich operator $S_{M_I/L_I} : TN \times M_I/L_I \to T(M_I/L_I)$ given by

$$S_{M_I/L_I}(x, \pi_I(z)) = T_z\pi_I\left(S(x,z)\right), \quad \text{for any } x \in N \text{ and } z \in M_I.$$

If Γ is a solution of the SDE with initial condition $\Gamma_0 \subset M_I$, then so is $\Gamma_{M_I/L_I} := \pi_I(\Gamma)$ with respect to S_{M_I/L_I} and X, with initial condition $\pi_I(\Gamma_0)$. S_{M_I/L_I} is the **reduced Stratonovich operator** *and Γ_{M_I/L_I} the* **reduced solution**.

Reconstruction

We now carry out the reverse procedure: given an isotropy subgroup $I \subset G$ and a solution semimartingale Γ_{M_I/L_I} of the reduced stochastic differential equation with Stratonovich operator S_{M_I/L_I} we will *reconstruct* a solution Γ of the initial stochastic differential equation with Stratonovich operator S.

Simplification: assume, without loss of generality, that the action is free. The general case can be obtained by replacing in the following paragraphs M by the isotropy type manifolds M_I, and G by the groups L_I.

Goal: Given any solution $\Gamma_{M/G}$ of the reduced system, write a corresponding solution Γ of the original system as $\Gamma = \Phi_{g^{\Xi}}(d)$ where $d : \mathbb{R}_+ \times \Omega \longrightarrow M$ is a semimartingale such that $\pi(d) = \Gamma_{M/G}$ and $g^{\Xi} : \mathbb{R}_+ \times \Omega \longrightarrow G$ is a G-valued semimartingale which satisfies a suitable stochastic differential equation on the group G.

Main tool: $A \in \Omega^1(M; \mathfrak{g})$ an auxiliary principal connection on the left principal G-bundle $\pi : M \longrightarrow M/G$.

$$TM = \mathrm{Hor} \oplus \mathrm{Ver} \quad T^*M = \mathrm{Hor}^* \oplus \mathrm{Ver}^*,$$

where, by definition, $\mathrm{Hor}^*_z := (\mathrm{Ver}_z)^\circ$ and $\mathrm{Ver}^*_z := (\mathrm{Hor}_z)^\circ$. Any one form $\alpha \in \Omega(M)$ may be uniquely written as

$$\alpha = \alpha^H + \alpha^V$$

with $\alpha^H \in \mathrm{Hor}^*$ and $\alpha^V \in \mathrm{Ver}^*$.

Horizontal lift of the reduced semimartingale

$\Gamma_{M/G} \subset M_{M/G}$ a solution of the reduced stochastic differential equation associated to $S_{M/G}$, with stochastic component $X : \mathbb{R}_+ \times \Omega \to V$.

A **horizontal lift** of $\Gamma_{M/G}$ (see Shigekawa, Catuogno) is a M-valued semimartingale $d : \mathbb{R}_+ \times \Omega \to M$ such that $d_0 = \Gamma_0$, $\pi(d) = \Gamma_{M/G}$ and that satisfies

$$\int \langle A, \delta d \rangle = 0, \tag{7}$$

where (7) is a $\mathfrak{g}$-valued integral. More specifically, let $\{\xi_1, ..., \xi_m\}$ be a basis of the Lie algebra $\mathfrak{g}$ and let $A(z) = \sum_{i=1}^m A^i(z)\,\xi_i$. Then

$$\int \langle A, \delta d \rangle := \sum_{i=1}^m \int \langle A^i, \delta d \rangle \xi_i. \tag{8}$$

Construction of the phase

It is easy to see that

$$\ker\left(T_g^*\Phi_z\right) = \left(T_{g\cdot z}\left(G\cdot z\right)\right)^\circ = \left(\mathrm{Ver}_{g\cdot z}\right)^\circ = \mathrm{Hor}_{g\cdot z}^*. \tag{9}$$

Therefore, the map

$$\widetilde{T_g^*\Phi_z} := T_g^*\Phi_z\Big|_{\mathrm{Ver}_{g\cdot z}^*} : T_{g\cdot z}^*M \cap \mathrm{Ver}_{g\cdot z}^* \longrightarrow T_g^*G \tag{10}$$

is an isomorphism. Let

$$\begin{array}{rccl} \rho\left(g,z\right): T_g^*G & \longrightarrow & T_{g\cdot z}^*M \cap \mathrm{Ver}_{g\cdot z}^* \subset T_{g\cdot z}^*M \\ \alpha_g & \longmapsto & \left(\widetilde{T_g^*\Phi_z}\right)^{-1}\left(\alpha_g\right) \end{array}$$

and define $\psi^*\left(x,z,g\right): T_g^*G \to T_x^*N$ by

$$\psi^*\left(x,z,g\right) = S^*\left(x,g\cdot z\right)\circ\rho\left(g,z\right).$$

We define a dual Stratonovich operator between the manifolds G and $M \times N$ as

$$\begin{array}{rcl} K^{*}\left(\left(z,x\right),g\right): T_{g}^{*}G & \longrightarrow & T_{z}^{*}M \times T_{x}^{*}N \\ \alpha_{g} & \longmapsto & \left(0, \psi^{*}\left(x,z,g\right)\left(\alpha_{g}\right)\right). \end{array} \tag{11}$$

The reconstruction theorem

Theorem

$X : \mathbb{R}_+ \times \Omega \to N$ a N-valued semimartingale and $S : TN \times M \to TM$ a Stratonovich operator invariant with respect to a free and proper action of G. $\Gamma_{M/G}$ a solution semimartingale of the reduced stochastic differential equation then $\Gamma = g^{\Xi} \cdot d$ is a solution of the original stochastic differential equation such that $\pi(\Gamma) = \Gamma_{M/G}$.

- $d : \mathbb{R}_+ \times \Omega \to M$ is the horizontal lift of $\Gamma_{M/G}$.
- $g^{\Xi} : \mathbb{R}_+ \times \Omega \to G$ is the semimartingale solution of the SDE $\delta g^{\Xi} = K(\Xi, g)\, \delta\Xi$ with initial condition $g_0^{\Xi} = e$, K the Stratonovich operator introduced in (11), and stochastic component $\Xi = (d, X)$.

d is the **horizontal lift** of $\Gamma_{M/G}$ and $\Gamma = g^{\Xi}$ the **stochastic phase** of the reconstructed solution.

- The skew-product decomposition of second order differential operators is a factorization technique that has been used in the stochastic processes literature in order to split the semielliptic and, in particular, the diffusion operators, associated to certain SDEs.
- This splitting has important consequences as to the properties of the solutions of these equations, like certain factorization properties of their probability laws and of the associated stochastic flows.
- Symmetries are a natural way to obtain this kind of decompositions.

Definition

N, M_1, and M_2 three smooth manifolds and

$$S(x,m) : T_xN \to T_m(M_1 \times M_2), \; x \in N \; m = (m_1, m_2) \in M_1 \times M_2,$$

a Stratonovich operator from N to the manifold $M_1 \times M_2$.
S admits a **skew-product decomposition** if there exists a Stratonovich operator $S_2(x, m_2) : T_xN \longrightarrow T_{m_2}M_2$ from N to M_2 and a M_2-dependent Stratonovich operator $S_1(x, m_1, m_2) : T_xN \to T_{m_1}M_1$ such that

$$S(x,m) = (S_1(x, m_1, m_2), S_2(x, m_2)) \in \mathcal{L}(T_xN, T_{m_1}M_1 \times T_{m_2}M_2)$$

for any $m = (m_1, m_2) \in M_1 \times M_2$. The operators S_1 and S_2 will be called the **factors** of S.

Skew product of a second order differential operator

Given a second order differential operator $L \in \mathfrak{X}_2(M_1 \times M_2)$ on $M_1 \times M_2$. A skew-product decomposition of L are two smooth maps $L_1 : M_2 \to \mathfrak{X}_2(M_1)$ and $L_2 \in \mathfrak{X}_2(M_2)$ such that for any $f \in C^\infty(M_1 \times M_2)$

$$L[f](m_1, m_2) = (L_1(m_2)[f(\cdot, m_2)])(m_1) + (L_2[f(m_1, \cdot)])(m_2).$$

If a Stratonovich operator associated to a semielliptic diffusion admits a skew-product decomposition then the corresponding Schwartz operator admits a skew-product decomposition, which in turn implies the availability of a skew-product decomposition of its infinitesimal generator.

The tangent-normal decomposition for ODEs

G acting properly on M via $\Phi : G \times M \longrightarrow M$ and X a G-equivariant vector field defined on a G-invariant open subset of M. Let m be a point in the domain of X and S a slice at m. Then there exist two vector fields X_T and X_N such that:

(i) $X_T \in \mathfrak{X}(G \cdot S)^G$ and $X_T(z) = (\xi(z))_M(z)$, $z \in G \cdot S$, where $\xi : G \cdot S \longrightarrow \mathfrak{g}$ is a smooth G-equivariant map such that $\xi(z) \in \operatorname{Lie} N(G_z) \cdot z$, for all $z \in G \cdot S$. Moreover, the flow F_t of X_T is given by $F_t(z) = \exp t\xi(z) \cdot z$.

(ii) $X_N \in \mathfrak{X}(S)^{G_m}$.

(iii) If $z = g \cdot s \in G \cdot S$ with $g \in G$ and $s \in S$, then

$$X(z) = X_T(z) + T_s\Phi_g \cdot X_N(s) = T_s\Phi_g \cdot (X_T(s) + X_N(s)).$$

(iv) If G_t is the flow of X_N, then the integral curve of X through the point $g \cdot s \in G \cdot S$ is

$$F_t(g \cdot s) = g(t) \cdot G_t(s)$$

where $g(t)$ is the solution of the first order differential equation

$$\dot{g}(t) = T_e L_{g(t)} \cdot \xi\left(G_t(s)\right), \quad g(0) = g.$$

The tangent-normal decomposition for SDEs

Let $X : \mathbb{R}_+ \times \Omega \to N$ be a N-valued semimartingale, $\Phi : G \times M \to M$ a proper Lie group action, and $S : TN \times M \to TM$ a G-invariant Stratonovich operator. Let $m \in M$ and W a slice at m. Then, there exist two Stratonovich operators $S_N : TN \times W \to TW$ and $S_T : TN \times G \cdot W \to T(G \cdot W)$ such that the following statements hold:

(i) The Stratonovich operator S_T is G-invariant and $S_T(x, z) \in \mathcal{L}\left(T_x N, \mathrm{Lie}\left(N(G_z)\right) \cdot z\right)$ for any $x \in N$ and any $z \in G \cdot W$. There exists an adjoint G-equivariant map $\xi : TN \times G \cdot W \to \mathfrak{g}$, $(\xi\left(x, g \cdot z\right) = \mathrm{Ad}_g \circ \xi\left(x, z\right))$ such that $S_T(x, z) = T_e \Phi_z \circ \xi\left(x, z\right)$.

(ii) The Stratonovich operator $S_N : TN \times W \to TW$ is G_m-invariant.

(iii) If $z = g \cdot w \in G \cdot W$, with $g \in G$ and $w \in W$, then

$$S(x, z) = S_T(x, z) + T_w\Phi_g \circ S_N(x, w) = T_w\Phi_g \circ (S_T(x, w) + S_N(x, w)).$$

This is the **tangent-normal decomposition** of S.

(iv) Let φ be the flow of the stochastic system (W, S_N, X, N) so that $\varphi(w)$ denotes the solution of

$$\delta\Gamma = S_N(X, \Gamma)\delta X \tag{12}$$

with initial condition $\Gamma_{t=0} = w$ a.s.. Let $S_{\mathfrak{g}\times W} : TN \times (\mathfrak{g} \times W) \to T(\mathfrak{g} \times W)$ be the Stratonovich operator $S_{\mathfrak{g}\times W}(x, (\eta, w)) = \xi(x, w) \times S_N(x, w) \in \mathcal{L}(T_xN, \mathfrak{g} \times T_wW)$ and (η^w, Γ^w) be the solution semimartingale of the stochastic system $(\mathfrak{g} \times W, S_{\mathfrak{g}\times W}, X, N)$ with initial condition $(0, w) \in \mathfrak{g} \times W$. Finally, let $\widetilde{g} : \{0 \leq t < \tau_\varphi\} \to G$ be the solution of the stochastic system $(G, L, \eta^w, \mathfrak{g})$ with initial condition $g \in G$ and where $L : T\mathfrak{g} \times G \to TG$ is such that $L(\eta, g)(\nu) = T_eL_g(\nu)$. Then, the semimartingale

$$\Gamma_t = \widetilde{g}_t \cdot \varphi_t(w)$$

is a solution up to time τ_φ of the stochastic system (M, S, X, N) with initial condition $z = g \cdot w \in G \cdot W$.

(v) Suppose that $G_w = G_z$, for any $w \in W$. Then S admits a local skew-product decomposition. More specifically, for any point $m \in M$, there exists an open neighborhood $V \subseteq G/G_m$ of G_m, a diffeomorphism $F : V \times W \to U \subseteq M$, and a skew-product split Stratonovich operator $S_{V\times W} : TN \times (V \times W) \to T(V \times W)$ such that F establishes a bijection between semimartingales Γ starting on U which are solution of the stochastic system (U, S, X, N) and semimartingales on $V \times W$ solution of the stochastic system $(V \times W, S_{V\times W}, X, N)$. Moreover,

$$S_{V\times W}(x, (gG_m, w)) = T_g\pi_{G_m} \circ T_e L_g (\xi(x, w)) \times S_N(x, w)$$

for any $x \in N$, $gG_m \in V \subset G/G_m$, and any $g \in G$ such that $\pi_{G_m}(g) = gG_m$.

The last shows that proper symmetries of Stratonovich operators imply the availability of skew-products decompositions around most points in the manifold where the solutions take place. Indeed, the Principal Orbit Type Theorem shows that there exists an isotropy subgroup H whose associated isotropy type manifold $M_{(H)} := \{z \in M \mid G_z = kHk^{-1}, k \in G\}$ is open and dense in M. Hence, for any point $m \in M_{(H)}$ there exist slice coordinates around the orbit $G \cdot m$ in which the manifold M looks locally like $G \times_H W = G \times_H W_H \simeq G/H \times W_H$. This local trivialization of the manifold M into two factors and the results in part **(v)** of the theorem can be used to split the Stratonovich operator S, in order to obtain a locally defined skew-product around all the points in the open dense subset $M_{(H)}$ of M.

Hamiltonian case: conservation laws

Proposition

Let (M, h, X, V) Hamiltonian system acted properly and canonically upon by a Lie group G. $h : M \to V^*$ is a G-invariant function.

(i) **Law of conservation of the isotropy**: The isotropy type submanifolds M_I are locally invariant submanifolds, for any isotropy subgroup $I \subset G$.

(ii) **Noether's Theorem**: If the G-action on $(M, \{\cdot, \cdot\})$ has a momentum map associated $\mathbf{J} : M \to \mathfrak{g}^*$ then its level sets are left invariant.

(iii) **Optimal Noether's Theorem**: The level sets of the optimal momentum map $\mathcal{J} : M \to M/D_G$ are locally invariant.

Reduction and reconstruction

$(M, \{\cdot,\cdot\}, h : M \rightarrow V^*, X)$ invariant with respect to the canonical, free , and proper G-action $\Phi : G \times M \rightarrow M$.

(i) Poisson reduction: The projection $h_{M/G}$ of h onto M/G, determined by $h_{M/G} \circ \pi = h$, with $\pi : M \rightarrow M/G$ the orbit projection, induces a Hamiltonian system on $(M/G, \{\cdot,\cdot\}_{M/G})$ with stochastic component X and whose Stratonovich operator $H_{M/G} : TV \times M/G \rightarrow T(M/G)$ is

$$H_{M/G}(v, \pi(z))(u) = T_z\pi\left(H(v,z)(u)\right) = \sum_{i=1}^{r} \langle \epsilon^i, u\rangle X_{h_i^{M/G}}(\pi(z))$$

The functions $h_i^{M/G} \in C^\infty(M/G)$ are the projections of $h_i \in C^\infty(M)^G$. Moreover, if Γ is a solution associated to H with initial condition Γ_0, then so is $\Gamma_{M/G} := \pi(\Gamma)$ with respect to $H_{M/G}$, with initial condition $\pi(\Gamma_0)$.

(ii) Symplectic reduction: Suppose that M is symplectic and that the action has a coadjoint equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ associated. Then for any $\mu \in \mathfrak{g}^*$, the function $h_\mu : M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu \to V^*$ uniquely determined by the equality $h_\mu \circ \pi_\mu = h \circ i_\mu$, induces a Hamiltonian system on the reduced space $(M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu, \omega_\mu)$ whose Stratonovich operator $H_\mu : TV \times M_\mu \to TM_\mu$ is given by

$$H_\mu(v, \pi_\mu(z))(u) = T_z\pi_\mu \left(H(v, i_\mu(z))(u)\right) = \sum_{i=1}^{r} \langle \epsilon^i, u \rangle X_{h_i^\mu}(\pi_\mu(z)).$$

The functions $h_i^\mu \in C^\infty(\mathbf{J}^{-1}(\mu)/G_\mu)$ are the coefficient functions in the linear combination $h_\mu = \sum_{i=1}^{r} h_i^\mu \epsilon^i$ and are related to the components $h_i \in C^\infty(M)^G$ of h via the relation $h_i^\mu \circ \pi_\mu = h_i \circ i_\mu$. Moreover, if Γ is a solution semimartingale of the Hamiltonian system associated to H with initial condition $\Gamma_0 \subset \mathbf{J}^{-1}(\mu)$, then so is $\Gamma_\mu := \pi_\mu(\Gamma)$ with respect to H_μ, with initial condition $\pi_\mu(\Gamma_0)$.

Stochastic collective Hamiltonian motion

This is a situation in which symplectic reduction not only cuts down dimension; it also makes the system deterministic. From the point of view of obtaining the solutions of the system, the procedures previously introduced allow in this case the splitting of the problem into two parts: first, the solution of a standard ordinary differential equation for the reduced system and second, the solution of a stochastic differential equation in the group at the time of the reconstruction.

A function of the form $f \circ \mathbf{J} \in C^\infty(M)$, for some $f \in C^\infty(\mathfrak{g}^*)$, is called **collective**. By the Collective Hamiltonian Theorem

$$X_{f \circ \mathbf{J}}(z) = \left(\frac{\delta f}{\delta \mu}\right)_M (z), \quad z \in M,\ \mu = \mathbf{J}(z),$$

where $\frac{\delta f}{\delta \mu} \in \mathfrak{g}$ is such that for any $\nu \in \mathfrak{g}^*$, $Df(\mu) \cdot \nu = \langle \nu, \frac{\delta f}{\delta \mu} \rangle$.

A consequence of this equality is that G-invariant functions commute with the collective functions. Indeed, if $h \in C^\infty(M)^G$, then for any $z \in M$,

$$\{h, f \circ \mathbf{J}\}(z) = \mathbf{d}h(z) \cdot X_{f \circ \mathbf{J}}(z) = \mathbf{d}h(z) \cdot \left(\frac{\delta f}{\delta \mu}\right)_M (z) = 0.$$

Let $Y : \mathbb{R}_+ \times \Omega \to \mathbb{R}^r$ be a $\mathbb{R}^r$-valued continuous semimartingale, $\{f_1, ..., f_r\} \subset C^\infty(\mathfrak{g}^*)$ a finite family of Ad^*_G-invariant functions on $\mathfrak{g}^*$, and $h_0 \in C^\infty(M)^G$. Consider the following G-invariant Hamiltonian

$$\begin{array}{rcl} h : M & \longrightarrow & \mathbb{R} \times \mathbb{R}^r \\ m & \longmapsto & \left(h_0(m), \left(f_1(\mathbf{J}(m)), ..., f_r(\mathbf{J}(m))\right)\right). \end{array}$$

Let X be the continuous semimartingale

$$\begin{array}{rcl} X : \mathbb{R}_+ \times \Omega & \longrightarrow & \mathbb{R}_+ \times \mathbb{R}^r \\ (t, \omega) & \longmapsto & (t, Y_t(\omega)). \end{array}$$

Consider the G-invariant stochastic Hamiltonian system (M, ω, h, X). Noether's theorem guarantees that the level sets of $\mathbf{J}$ are left invariant. The reduced stochastic Hamiltonian system $(M_\mu, \omega_\mu, h_\mu, X)$ is such that

$$h_\mu \circ \pi_\mu = h_0 \circ i_\mu,$$

since $\mathbf{J}$, and hence the functions $f_i \circ \mathbf{J}$, are constant on the level sets $\mathbf{J}^{-1}(\mu)$, for any $i = 1, ..., r$. Consequently, the reduced system $(M_\mu, \omega_\mu, h_\mu, X)$ is equivalent to the deterministic Hamiltonian system $(M_\mu, \omega_\mu, h_\mu)$. In other words, the reduced system obtained from (M, ω, h, X) coincides with the one obtained in deterministic mechanics by symplectic reduction of $(M, h_0, t, \mathbb{R}_+)$. Thus, *we have perturbed stochastically a symmetric mechanical system preserving its symmetries and without changing the deterministic behavior of its corresponding reduced system.*

Stochastic mechanics on Lie groups

G a Lie group, $X : \mathbb{R}_+ \times \Omega \to V$ a noise semimartingale, and $h : T^*G \to V^*$ invariant under the lifted left translations of G on T^*G. If we use body coordinates and we visualize T^*G as the product $G \times \mathfrak{g}^*$, the invariance of $h : G \times \mathfrak{g}^* \to V^*$ allows us to write it as $h = \sum_{i=1}^r h_i \epsilon^i$, $h_1, \ldots, h_r \in C^\infty(\mathfrak{g}^*)$. In body coordinates

$$\delta\Gamma^h = \sum_{i=1}^r \left(T_e L_{\Gamma^G} \left(\frac{\delta h_i}{\delta \Gamma^{\mathfrak{g}^*}} \right), \mathrm{ad}^*_{\frac{\delta h_i}{\delta \Gamma^{\mathfrak{g}^*}}} \Gamma^{\mathfrak{g}^*} \right) \delta X^i$$

where Γ^G and $\Gamma^{\mathfrak{g}^*}$ are the G and $\mathfrak{g}^*$ components of Γ^h, respectively, that is, $\Gamma^h := \left(\Gamma^G, \Gamma^{\mathfrak{g}^*}\right)$.

In the left trivialized representation, the reduced Poisson and symplectic Hamiltonians are simply the restrictions $h^{\mathfrak{g}^*}$ and $h^{\mathcal{O}_\mu}$ of h to $\mathfrak{g}^*$ and to the coadjoint orbits $\mathcal{O}_\mu \subset \mathfrak{g}^*$, respectively. Additionally, the reduced stochastic Hamilton equations on $\mathfrak{g}^*$ and $\mathcal{O}_\mu$ are given by

$$\delta \Gamma^{\mathfrak{g}^*} = \sum_{i=1}^{r} \mathrm{ad}^*_{\frac{\delta h_i^{\mathfrak{g}^*}}{\delta \Gamma^{\mathfrak{g}^*}}} \Gamma^{\mathfrak{g}^*} \delta X^i \quad \text{and} \quad \delta \Gamma^{\mathcal{O}_\mu} = \sum_{i=1}^{r} \mathrm{ad}^*_{\frac{\delta h_i^{\mathcal{O}_\mu}}{\delta \Gamma^{\mathcal{O}_\mu}}} \Gamma^{\mathcal{O}_\mu} \delta X^i \tag{13}$$

where $h^{\mathfrak{g}^*} = \sum_{i=1}^{r} h_i^{\mathfrak{g}^*} \epsilon^i$ and $h^{\mathcal{O}_\mu} = \sum_{i=1}^{r} h_i^{\mathcal{O}_\mu} \epsilon^i$.

In this setup, the dynamical reconstruction of reduced solutions is particularly simple to write down. Indeed, suppose that we are given a solution $\Gamma^{\mathfrak{g}^*}$ of, say, the Poisson reduced system. In order to obtain the solution Γ^h of the original system such that $\Gamma_0^h = (\Gamma_0^G, \Gamma_0^{\mathfrak{g}^*})$ and $\pi(\Gamma^h) = \Gamma^{\mathfrak{g}^*}$ it suffices to solve the stochastic differential equation in G

$$\delta\Gamma^G = \sum_{i=1}^{r} T_e L_{\Gamma^G}\left(\frac{\delta h_i}{\delta\Gamma^{\mathfrak{g}^*}}\right)\delta X^i, \tag{14}$$

with the initial condition Γ_0^G. The reconstructed solution that we are looking for is then $\Gamma^h = \left(\Gamma^G, \Gamma^{\mathfrak{g}^*}\right)$.

Stochastic perturbations of the free rigid body

The free rigid body, also referred to as Euler top, is a particular case of systems introduced in the previous section where the group G is $SO(3, \mathbb{R})$. We call it free because the energy of the system is purely kinetic and there is no potential term. Let $(\cdot, \cdot)$ be a left invariant Riemannian metric on G; the kinetic energy E associated to $(\cdot, \cdot)$ is $E(v) = \frac{1}{2}(v, v)$, $v \in TG$. Then, using the left invariance of the metric, we can write in body coordinates

$$E(g, \xi) = \frac{1}{2}(\xi, \xi)_e = \frac{1}{2}\langle I(\xi), \xi\rangle,$$

for any $(g, \xi) \in G \times \mathfrak{g}$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between elements of $\mathfrak{g}^*$ and $\mathfrak{g}$, and $I : \mathfrak{g} \to \mathfrak{g}^*$ is the map given by $\xi \longmapsto (\xi, \cdot)_e$ and usually known as the **inertia tensor** associated to the metric $(\cdot, \cdot)$.

The Legendre transformation associated to E can be used to define a Hamiltonian function $h: T^*G \longrightarrow \mathbb{R}$ that, in body coordinates, can be written as

$$h(g, \mu) = \frac{1}{2} \langle \mu, \Lambda(\mu) \rangle, \tag{15}$$

where $\Lambda = I^{-1}$. Notice that as the kinetic energy is left invariant (invariant with respect to the lifted G-action to T^*G of the action of G on itself by left translations), then the components of $\mathbf{J}_L$ are conserved quantities of the corresponding Hamiltonian system. Let $f \in C^\infty(\mathfrak{g}^*)$ be the function $f: \mathfrak{g}^* \longrightarrow \mathbb{R}$ given by $\mu \longmapsto \frac{1}{2} \langle \mu, \Lambda(\mu) \rangle$. The Hamiltonian function h may be expressed as $h = f \circ \mathbf{J}_R$. Therefore h is collective with respect to $\mathbf{J}_R$.

Back to the free rigid body case, that is, $G = SO(3, \mathbb{R})$. The Lie algebra $\mathfrak{so}(3, \mathbb{R})$ is the vector space of three dimensional skew-symmetric real matrices whose bracket is just the commutator of two matrices. As a Lie algebra, $(\mathfrak{so}(3), [\cdot, \cdot])$ is naturally isomorphic to $\left(\mathbb{R}^3, \times\right)$, where $\times$ denotes the cross product of vectors in $\mathbb{R}^3$. Under this isomorphism, the adjoint representation of $SO(3, \mathbb{R})$ on its Lie algebra is simply the action of matrices on vectors of $\mathbb{R}^3$ and the Lie-Poisson structure on $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is given by

$$\{f, g\}(v) = -v \cdot (\nabla f \times \nabla g),$$

for any $f, g \in C^\infty(\mathbb{R}^3)$, where ∇ is the usual Euclidean gradient. Given a free rigid body with inertia tensor $I : \mathbb{R}^3 \to \mathbb{R}^3$, since $\delta h_B/\delta\mu = \Lambda(\mu)$, for any $\mu \in \mathbb{R}^3$, the left-trivialized equations of motion of the system are

$$(\dot{A}, \dot{\mu}) = \left(A \cdot \widehat{\Lambda(\mu)}, \mu \times \Lambda(\mu)\right), \tag{16}$$

In the context of the free rigid body motion the momentum map $\mathbf{J}_L$ (respectively, $\mathbf{J}_R$) is called **patial angular momentum** (respectively, **body angular momentum**). The second component of (16), that is,

$$\dot{\mu} = \mu \times \Lambda(\mu) \tag{17}$$

are the well-known **Euler equations** for the free rigid body.

Random perturbations of the body angular momentum Let $V = \mathbb{R} \times \mathfrak{so}(3) \simeq \mathbb{R}^+ \times \mathbb{R}^3$ and let h be the Hamiltonian function $h : T^*SO(3) \to V^* = \mathbb{R} \times \mathfrak{so}(3)^*$ defined as $h = (h_0, \mathbf{J}_R)$, where h_0 is the Hamiltonian function of the free (deterministic) rigid body. Observe that h is a left-invariant function because so is $\mathbf{J}_R$. Let $Y : \mathbb{R}_+ \times \Omega \to \mathfrak{g}$ be a continuous semimartingale which we may suppose, for the sake of simplicity, is a $\mathfrak{g}$-valued Brownian motion and let $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}^* \times \mathfrak{g}$ be the semimartingale defined as $X_t(\omega) = (t, Y_t(\omega))$ for any $(t, \omega) \in \mathbb{R} \times \Omega$. Consider the stochastic Hamiltonian system on T^*G associated to h and X.

Since h is left invariant, the momentum map $\mathbf{J}_L$ is preserved by the solution semimartingales of this system and moreover, we can apply the reduction scheme introduced in the previous sections. If we carry out Poisson reduction we have a reduced Hamiltonian function $h^{\mathfrak{g}^*} : \mathfrak{g}^* \rightarrow V^*$ given by $h^{\mathfrak{g}^*}(\mu) = \left(\frac{1}{2}\langle \mu, \Lambda(\mu)\rangle, \mu\right)$. Let $\{\xi_1, \xi_2, \xi_3\}$ a basis of the Lie algebra $\mathfrak{g}$ and $\left\{\epsilon^1, \epsilon^2, \epsilon^3\right\} \subset \mathfrak{g}^*$ its dual basis. Observe that if we write $\mathbf{J}_R(\mu) = \sum_{i=1}^3 \langle \mu, \xi_i\rangle \epsilon^i$ and $Y = \sum_{i=1}^3 Y^i \xi_i$, then the reduced stochastic Lie-Poisson equations can be expressed as

$$\delta\mu_t = \mu_t \times \Lambda(\mu_t)\,\delta t + \sum_{i=1}^{3} (\mu_t \times \xi_i)\,\delta Y_t^i. \tag{18}$$

Regarding the reconstruction of the reduced dynamics, one has to solve the stochastic differential equation on the rotations group $SO(3)$ given by

$$\delta A_t = A_t \cdot \widehat{\Lambda(\mu_t)} \delta t + \sum_{i=1}^{3} A_t \cdot \widehat{\xi}_i \delta Y_t^i. \tag{19}$$

A physical model whose description fits well in a stochastic Hamiltonian differential equation like the one associated to h and X is that of a free rigid body subjected to small random impacts. Each impact causes a small and instantaneous change in the *body* angular momenta μ_t at time t that justifies the extra term in (18), when compared to the Euler equations (17).

Not so rigid rigid bodies; random perturbation of the inertia tensor

In this example we want to write the equations that describe a rigid body some of whose parts are slightly loose, that is, the body is not a true rigid body and hence its mass distribution is constantly changing in a random way. This will be modelled by stochastically perturbing the tensor of inertia.
For the sake of simplicity, we will write $G = SO(3, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{so}(3)$. Let $\mathcal{L}(\mathfrak{g}^*, \mathfrak{g})$ be the vector space of linear maps from $\mathfrak{g}^*$ to $\mathfrak{g}$. As we know $(\mathfrak{so}(3), [\cdot,\cdot]) \simeq \left(\mathbb{R}^3, \times\right)$. Furthermore, we can establish an isomorphism $\mathbb{R}^3 \simeq \left(\mathbb{R}^3\right)^*$ using the Euclidean inner product and hence we can write $\mathfrak{g} \simeq \mathfrak{g}^*$. Let $V = \mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g}) = \{A \in \mathcal{L}(\mathfrak{g}^*, \mathfrak{g}) \mid A^* = A\}$ be the vector space of selfadjoint linear maps from $\mathfrak{g}^*$ to $\mathfrak{g}$.

Define the Hamiltonian $h : T^*G \to V^*$ in *body coordinates* as

$$\begin{array}{rcl} h : T^*G \simeq G \times \mathfrak{g}^* & \longrightarrow & V^* \\ (g, \mu) & \longmapsto & \bar{\mu}, \end{array}$$

where $\bar{\mu}$ is such that

$$\begin{array}{rcl} \bar{\mu} : \mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g}) & \longrightarrow & \mathbb{R} \\ A & \longmapsto & \frac{1}{2}\langle \mu, A(\mu) \rangle . \end{array}$$

Observe that in body coordinates the Hamiltonian h does not depend on G, so the Hamiltonian is G-invariant by the action $\bar{\Phi}_L$ on T^*G. On the other hand, consider some filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, P\right)$ and introduce a stochastic component $X : \mathbb{R}_+ \times \Omega \to V$ in the following way:

$$\begin{array}{rcl} X : \mathbb{R}_+ \times \Omega & \longrightarrow & \mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g}) \\ (t, \omega) & \longmapsto & \Lambda t + \varepsilon A_t(\omega) . \end{array}$$

$\Lambda \in \mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g})$ plays the role of the inverse of the tensor of inertia given by the deterministic (rigid) description of the body, ε is a small parameter, and A is an arbitrary $\mathcal{L}_S(\mathfrak{g}^*, \mathfrak{g})$-valued semimartingale. In order to show how the stochastic Hamiltonian system on T^*G associated to h and X models a free rigid body whose inertia tensor undergoes random perturbations, we write down the associated stochastic reduced Lie-Poisson equations in Stratonovich form

$$\delta\mu_t = \mu_t \times \Lambda(\mu_t)\,\delta t + \varepsilon\mu_t \times \delta A_t(\mu_t).$$

Thus we see that these Lie-Poisson equations consist in changing $\Lambda(\mu_t)\,dt$ in the Euler equations (17) by $\Lambda(\mu_t)\,\delta t + \varepsilon\delta A_t(\mu_t)$, which accounts for the stochastic perturbation of the inertia tensor.