Applications of the inverse problem of Lagrangian mechanics

Tom Mestdag
Ghent University (Belgium)

Outline

1. **The inverse problem of Lagrangian mechanics**

2. **An inverse problem in the context of nonholonomic mechanics**

   [A.M. Bloch, O.E. Fernandez and T. Mestdag, Hamiltonization of nonholonomic systems and the inverse problem of the calculus of variations, to appear in *Reports on Mathematical Physics* (2008)]

3. **The inverse problem for invariant Lagrangians on a Lie group**

   [M. Crampin and T. Mestdag, The inverse problem for invariant Lagrangians on a Lie group, *Journal of Lie Theory* **18** (2008), 471–502]

These papers and other papers can be found on

http://users.ugent.be/$\sim$tmestdag

The inverse problem of Lagrangian mechanics

Problem. When are $\ddot{q}^i = f^i(q, \dot{q})$ equivalent to the Euler-Lagrange equations of a (yet to be determined) **regular Lagrangian** L?

⇝ Find multipliers $g_{ij}(q, \dot{q})$ such that $g_{ij}(\ddot{q}^j - f^j) = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^i}\right) - \dfrac{\partial L}{\partial q^i}$.

Solution. The multipliers must satisfy the **Helmholtz conditions**

$$\det(g_{ij}) \neq 0, \qquad g_{ji} = g_{ij}, \qquad \frac{\partial g_{ij}}{\partial \dot{q}^k} = \frac{\partial g_{ik}}{\partial \dot{q}^j},$$

$$\Gamma(g_{ij}) - \nabla^k_j g_{ik} - \nabla^k_i g_{kj} = 0, \qquad g_{ik}\Phi^k_j = g_{jk}\Phi^k_i$$

Here: $\Gamma = \dot{q}^i \partial/\partial q^i + f^i \partial/\partial \dot{q}^i$ is the corresponding VF on TQ, $\nabla^k_j = -\frac{1}{2}\partial_{\dot{q}^j} f^i$ and $\Phi^k_j = \Gamma\left(\partial_{\dot{q}^j} f^k\right) - 2\partial_{q^j} f^k - \frac{1}{2}\partial_{\dot{q}^j} f^l \partial_{\dot{q}^l} f^k$

Conversely: if the above has a solution, $\ddot{q}^i = f^i$ is derivable from a Lagrangian.

⇝ If a Lagrangian exists, then its Hessian w.r.t. fibre coord. $\dot{q}^i$ is a multiplier!

⇝ This is a mixed set of algebraic eq. and PDE for g_{ij} (in $2n$-variables $(q^i, \dot{q}^i)$).

⇝ The Helmholtz conditions are coordinate independent and can be cast in intrinsic form.

For some history on the inverse problem: see

O. Krupková, and G.E. Prince, Second-order ordinary differential equations in jet bundles and the inverse problem of the calculus of variations, Chapter 16 of D. Krupka and D. J. Saunders (eds.), *Handbook of Global Analysis*, Elsevier (2007), 837-904.

Method • Derive as many algebraic conditions as possible:

⇝ Start from the algebraic Φ-conditions $g_{ik}\Phi^k_j = g_{jk}\Phi^k_i$.

⇝ Take a $\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial \dot{q}^i}$-derivative of the Φ-condition, and replace $\Gamma(g_{ij})$ everywhere by means of the PDE ∇-condition.

We get $g_{ik}(\nabla\Phi)^k_j = g_{jk}(\nabla\Phi)^k_i$, where

$(\nabla\Phi)^i_j = \frac{d}{dt}(\Phi^i_j) - \nabla^i_m \Phi^m_j - \nabla^m_j \Phi^i_m$.

⇝ Repeat the above process on the $(\nabla\Phi)$-condition, and so on to obtain possibly independent $(\nabla \ldots \nabla\Phi)$-conditions.

⇝ Take derivatives of the Φ-equation in $\dot{q}$-directions and sum them up in such a way that the terms in $\partial_{\dot{q}^k} g_{ij}$ disappear on account of the symmetry. We get: $g_{ij}R^j_{kl} + g_{lj}R^j_{ik} + g_{kj}R^j_{li} = 0$, where $R^j_{kl} = \partial_{\dot{q}^j}(\Phi^k_i) - \partial_{\dot{q}^i}(\Phi^k_j)$.

• Once we have used up all the information that we can obtain from this infinite series of algebraic conditions, we can start looking at the PDE ∇-conditions.

Example 1

Does a regular Lagrangian exist for $\ddot{x}=0,\ \ddot{y}=-\dfrac{x}{1+x^2}\dot{x}\dot{y},\ \ddot{z}=-\dfrac{1}{1+x^2}\dot{x}\dot{y}$?

Non-vanishing components of Φ and $\nabla\Phi$ are

$\Phi^2_1=-\frac{x^2-2}{2(1+x^2)^2}\dot{x}\dot{y},\quad \Phi^2_2=\frac{x^2-2}{2(1+x^2)^2}\dot{x}^2,\quad \Phi^3_1=-\frac{3x}{2(1+x^2)^2}\dot{x}\dot{y},\quad \Phi^3_2=\frac{3x}{2(1+x^2)^2}\dot{x}^2.$

$(\nabla\Phi)^2_1=\frac{x(x^2-5)}{(1+x^2)^3}\dot{x}^2\dot{y},\ (\nabla\Phi)^2_2=-\frac{x(x^2-5)}{(1+x^2)^3}\dot{x}^3,\ (\nabla\Phi)^3_1=\frac{5x^2-1}{(1+x^2)^3}\dot{x}^2\dot{y},\ (\nabla\Phi)^3_2=-\frac{5x^2-1}{(1+x^2)^3}\dot{x}^3.$

The (1,3) Φ and $\nabla\Phi$-equations are:

$$\begin{aligned}
g_{11}\Phi^1_3+g_{12}\Phi^2_3+g_{13}\Phi^3_3 &= g_{31}\Phi^1_1+g_{32}\Phi^2_1+g_{33}\Phi^3_1\\
\Leftrightarrow\quad 0 &= (x^2-2)g_{23}+3xg_{33}\\
g_{11}(\nabla\Phi)^1_3+g_{12}(\nabla\Phi)^2_3+g_{13}(\nabla\Phi)^3_3 &= g_{31}(\nabla\Phi)^1_1+g_{32}(\nabla\Phi)^2_1+g_{33}(\nabla\Phi)^3_1\\
\Leftrightarrow\quad 0 &= (x^3-5x)g_{23}+(5x^2-1)g_{33}.
\end{aligned}$$

We have $\det=2x^4+4x^2+2$, so $g_{23}=g_{33}=0$.

With that the (1,2) Φ and $\nabla\Phi$-equations are:

$$\begin{aligned} g_{11}\Phi^1_2 + g_{12}\Phi^2_2 + g_{13}\Phi^3_2 &= g_{21}\Phi^1_1 + g_{22}\Phi^2_1 + g_{23}\Phi^3_1 \\ \Leftrightarrow \quad (x^2-2)\dot{x}g_{12} + 3x\dot{x}g_{13} &= -(x^2-2)\dot{y}g_{22} \\ g_{11}(\nabla\Phi)^1_2 + g_{12}(\nabla\Phi)^2_2 + g_{13}(\nabla\Phi)^3_2 &= g_{21}(\nabla\Phi)^1_1 + g_{22}(\nabla\Phi)^2_1 + g_{23}(\nabla\Phi)^3_1 \\ \Leftrightarrow \quad (x^3-5x)\dot{x}g_{12} + (5x^2-1)\dot{x}g_{13} &= -(x^3-5x)\dot{y}g_{22}. \end{aligned}$$

We have $\det = 2x^4 + 4x^2 + 2$, so $g_{13} = 0$ and $\dot{x}g_{12} = -\dot{y}g_{22}$.

$\rightsquigarrow$ There is no regular Lagrangian!

Example 2

The **harmonic oscillator** in 2 dimensions: $\ddot{x} = -x$, $\ddot{y} = -y$.

$\rightsquigarrow$ A Lagrangian is $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(x^2 + y^2)$.

Helmholtz conditions? • We have $\Phi^1_1 = \Phi^2_2 = 2$, $\Phi^1_2 = \Phi^2_1 = 0$ and $\nabla\Phi = 0$.

$\rightsquigarrow$ The Φ-equation $g_{11}\Phi^1_2 + g_{12}\Phi^2_2 = g_{21}\Phi^1_1 + g_{22}\Phi^2$ gives $0 = 0$!

• Also $\nabla^j_i = 0 \rightsquigarrow$ the ∇-equations are: $\Gamma(g_{ij}) = 0$.

So, among poss. other, all $g = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ (with $a, b, c \in \mathbf{R}$, $\det g \neq 0$) are ok!

$\rightsquigarrow$ A function whose Hessian is g (up to linear terms in $\dot{x}$ and $\dot{y}$) is of the form:

$$L = \frac{1}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) + f(x, y);$$

When is this a Lagrangian? If $\Gamma(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i} = 0$, or

$$\frac{\partial f}{\partial x} = -ax - by \quad \text{and} \quad \frac{\partial f}{\partial y} = -bx - cy \quad \Rightarrow \quad f = -1/2(ax^2 + 2bxy + cy^2) + d.$$

$\rightsquigarrow$ Take $a = c = 0, b = 1$: also $L = \dot{x}\dot{y} - xy$ is a Lagrangian!

Hamiltonization of nonholonomic systems

Motivation. M. Abud Filho, L.C. Gomes, F.R.A. Simao and F.A.B. Coutinho, The Quantization of Classical Non-holonomic Systems, *Revista Brasileira de Fisica* **13** (1983) 384-406.

$\rightsquigarrow$ For some of the well-known classical examples, they propose a Hamiltonian, whose Hamilton equations, when restricted to a certain subset of phase space, **reproduce the nonholonomic dynamics**.

$\rightsquigarrow$ How? They start off from the actual **solutions** of the nonholonomic system, and apply a sort of Hamilton-Jacobi theory to arrive at the Hamiltonian.

$\rightsquigarrow$?? What if the explicit solutions of the system are **not readily available** ??

A different method in this talk. Look for a regular **Lagrangian** first and use the **Legendre transformation** to obtain a Hamiltonian with the above property.

$\rightsquigarrow$ The constraint distribution will be mapped onto a constraint submanifold in phase space. The nonholonomic solutions, considered as particular solutions of the Hamiltonian system, will then all lie on that submanifold.

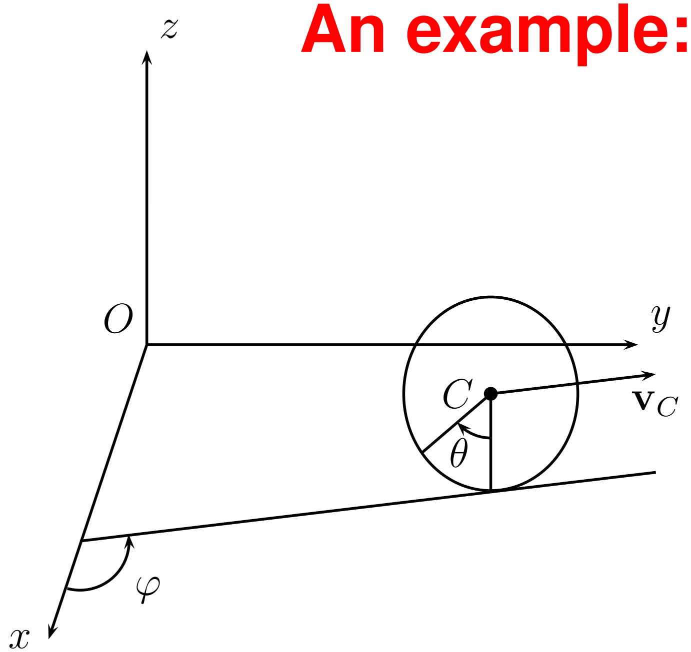
The Euler-Lagrange equations of a regular Lagrangian are **second-order differential equations**.

1. How can we **associate a second-order systems** to (a class of) nonholonomic systems?

2. How can we **find a Lagrangian** from a given second-order system?

$\rightsquigarrow$ Use the Helmholtz conditions of the IP!

An example: The vertically rolling disk



Coordinates:

- (x, y): centre of mass C,
- φ: angle of the disk with the (x, z)-plane,
- θ: angle of a fixed line on the disk and a vertical line.

$(\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z)$ follows the disk. $\rightsquigarrow \boldsymbol{\omega} = \dot\theta \mathbf{e}_\varphi + \dot\varphi \mathbf{e}_z = -\dot\theta \sin\varphi \mathbf{e}_x + \dot\theta \cos\varphi \mathbf{e}_y + \dot\varphi \mathbf{e}_z$

Constraints: $z = R$ (holonomic), rolling without slipping (non-holonomic):

$$\mathbf{v}_A = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{AC} \quad \Leftrightarrow \quad \left\{ \begin{array}{rcl} \dot x & = & R\cos\varphi\,\dot\theta \\ \dot y & = & R\sin\varphi\,\dot\theta \end{array} \right.$$

The Lagrangian is $L = T = \frac{1}{2} M \mathbf{v}_C^2 + \frac{1}{2} I_C(\boldsymbol{\omega}, \boldsymbol{\omega})$

$$= \tfrac{1}{2} M(\dot x^2 + \dot y^2) + \tfrac{1}{2} I \dot\theta^2 + \tfrac{1}{2} J \dot\varphi^2, \ \ (I = \tfrac{1}{2} M R^2, J = \tfrac{1}{4} M R^2).$$

Next to the constraints, the equations of motion are

$$\left\{\begin{array}{l}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)-\frac{\partial T}{\partial x}=\lambda_1,\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right)-\frac{\partial T}{\partial y}=\lambda_2,\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}=-\lambda_1 R\cos\varphi-\lambda_2 R\sin\varphi,\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right)-\frac{\partial T}{\partial \varphi}=0,\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}M\ddot{x}=\lambda_1,\\ M\ddot{y}=\lambda_2,\\ I\ddot{\theta}=-\lambda_1 R\cos\varphi-\lambda_2 R\sin\varphi,\\ J\ddot{\varphi}=0,\end{array}\right.$$

$\rightsquigarrow$ Use the constraints $\dot{x}=R\cos\varphi\,\dot{\theta},\ \dot{y}=R\sin\varphi\,\dot{\theta}$ to eliminate λ_1 and λ_2:

$$\ddot{\theta}=0,\quad \ddot{\varphi}=0,\quad \dot{x}=R\cos(\varphi)\dot{\theta},\quad \dot{y}=R\sin(\varphi)\dot{\theta}.$$

$\rightsquigarrow$ The solutions are $\theta(t)=u_\theta t+\theta_0,\ \varphi(t)=u_\varphi t+\varphi_0$, and

- if $u_\varphi\neq 0$: $x(t)=\left(\frac{u_\theta}{u_\phi}\right)R\sin(\varphi(t))+x_0,\ y(t)=-\left(\frac{u_\theta}{u_\phi}\right)R\cos(\varphi(t))+y_0$.
- If $u_\varphi=0$: $x(t)=R\cos(\varphi_0)u_\theta t+x_0,\quad y(t)=R\sin(\varphi_0)u_\theta t+y_0$.

Associated second-order systems

The dynamics is a mixed set of coupled first- and second-order eq.:

$$\ddot{\theta} = 0, \quad \ddot{\varphi} = 0, \quad \dot{x} = R\cos(\varphi)\dot{\theta}, \quad \dot{y} = R\sin(\varphi)\dot{\theta}.$$

$\leadsto$ There are, however, infinitely many systems of “associated” second-order equations (only), whose solution set **contains** the solutions of the nonhol. eq.

Examples. 1. the *‘first’ associated second-order system*:

$$\ddot{\theta} = 0, \quad \ddot{\varphi} = 0, \quad \ddot{x} = -R\sin(\varphi)\dot{\theta}\dot{\varphi}, \quad \ddot{y} = R\cos(\varphi)\dot{\theta}\dot{\varphi}.$$

$\leadsto$ If $u_\varphi \neq 0$, its solutions are $\theta(t) = u_\theta t + \theta_0,\ \varphi(t) = u_\varphi t + \varphi_0$

$$x(t) = \left(\frac{u_\theta}{u_\varphi}\right) R\sin(\varphi(t)) + u_x t + x_0,$$

$$y(t) = -\left(\frac{u_\theta}{u_\varphi}\right) R\cos(\varphi(t)) + u_y t + y_0.$$

$\leadsto$ By restricting to those for which $\dot{x} = \cos(\varphi)\dot{\theta}$ and $\dot{y} = \sin(\varphi)\dot{\theta}$ (i.e. $u_x = u_y = 0$), we get back the solutions of the non-holonomic eq. (and similarly for solutions with $u_\varphi = 0$).

The constraints are $\dot{x} = R\cos(\varphi)\dot{\theta}$ and $\dot{y} = R\sin(\varphi)\dot{\theta}$

1. the *'first' associated second-order system*:

$$\ddot{\theta} = 0, \quad \ddot{\varphi} = 0, \quad \ddot{x} = -R\sin(\varphi)\dot{\theta}\dot{\varphi}, \quad \ddot{y} = R\cos(\varphi)\dot{\theta}\dot{\varphi}.$$

2. the *'second' associated second-order system*:

$$\ddot{\theta} = 0, \quad \ddot{\varphi} = 0, \quad \ddot{x} = -\frac{\sin(\varphi)}{\cos(\varphi)}\dot{x}\dot{\varphi}, \quad \ddot{y} = \frac{\cos(\varphi)}{\sin(\varphi)}\dot{y}\dot{\varphi}.$$

3. Another associated system is: $\ddot{\theta} = 0,\ \ddot{\varphi} = 0,\ \ddot{x} = -\dot{y}\dot{\varphi},\ \ddot{y} = \dot{x}\dot{\varphi}.$

4. Given that $\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y} = 0$,

$$\begin{aligned}
J\ddot{\varphi} &= -mR(\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y})\dot{\theta}, \\
(I + mR^2)\ddot{\theta} &= mR(\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y})\dot{\varphi}, \\
(I + mR^2)\ddot{x} &= -R(I + mR^2)\sin(\varphi)\dot{\theta}\dot{\varphi} + mR^2\cos(\varphi)(\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y})\dot{\varphi}, \\
(I + mR^2)\ddot{y} &= R(I + mR^2)\cos(\varphi)\dot{\theta}\dot{\varphi} + mR^2\sin(\varphi)(\sin(\varphi)\dot{x} - \cos(\varphi)\dot{y})\dot{\varphi}.
\end{aligned}$$

5. Many more ....

Are these associated second-order equations equivalent to the Euler-Lagrange equations of some regular Lagrangian or not?

$\Big($ Answer for 4 from [FernandezBloch08]:

$L = -\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 + mR\dot{\theta}(\cos(\varphi)\dot{x} + \sin(\varphi)\dot{y})\Big)$.

A class of nonholonomic systems

Problems. $\rightsquigarrow$ There is **no systematic way** to catalogue the second-order systems that are associated to a nonholonomic system.

$\rightsquigarrow$ If no regular Lagrangian exists for one assoc. system, it may still exist for one of the infinitely many other assoc. systems.

$\rightsquigarrow$ The solution of the inverse problem is too hard and too technical to tackle in full generality.

$\Rightarrow$ We will chose a class of nonholonomic systems.

Assumptions.
- Q is locally just $\mathbb{R}^n$ with coordinates $(r_1, r_2; s_\alpha)$.
- the **Lagrangian** is given by $L = \frac{1}{2}(I_1\dot{r}_1^2 + I_2\dot{r}_2^2 + \sum_\alpha I_\alpha \dot{s}_\alpha^2)$.
- the **constraints** take the form $\dot{s}_\alpha = -A_\alpha(r_1)\dot{r}_2$.

Examples.
- Vertically rolling disk.
- a nonholonomically constrained free particle

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right), \qquad \dot{z} + x\dot{y} = 0.$$

- the knife edge on a plane

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\phi}^2, \qquad \dot{x}\sin(\phi) - \dot{y}\cos(\phi) = 0.$$

Nonholonomic Eq. $\ddot{r}_1 = 0, \quad \ddot{r}_2 = -N^2 K \dot{r}_1 \dot{r}_2, \quad \dot{s}_\alpha = -A_\alpha \dot{r}_2,$

where $N(r_1) = \dfrac{1}{\sqrt{I_2 + \sum_\alpha I_\alpha A_\alpha^2}}$ (related to the invariant measure)

$K = \sum_\beta I_\beta A_\beta A'_\beta$.

$\rightsquigarrow$ In case of the vertically rolling disk N is a constant and $K = 0 \Rightarrow \ddot{r}_2 = 0$.

Associated systems of interest.

- The **first** associated second-order system is

$$\ddot{r}_1 = 0, \quad \ddot{r}_2 = -N^2 K \dot{r}_1 \dot{r}_2, \quad \ddot{s}_\alpha = -\Big(A'_\alpha - N^2 K A_\alpha\Big)\dot{r}_1 \dot{r}_2.$$

$\rightsquigarrow$ It is of the form $\ddot{r}_1 = 0, \quad \ddot{r}_2 = \Gamma_2(r_1)\dot{r}_1\dot{r}_2, \quad \ddot{s}_\alpha = \Gamma_\alpha(r_1)\dot{r}_1\dot{r}_2.$

- The **second** associated second-order system is

$$\ddot{r}_1 = 0, \quad \ddot{r}_2 = -N^2 K \dot{r}_1 \dot{r}_2, \quad \ddot{s}_\alpha = \Big(A'_\alpha - N^2 K A_\alpha\Big)\dot{r}_1 \left(\frac{\dot{s}_\alpha}{A_\alpha}\right)$$

(no sum over α).

$\rightsquigarrow$ It is of the form $\ddot{r}_1 = 0, \quad \ddot{q}_a = \Xi_a(r_1)\dot{q}_a \dot{r}_1$

(no sum over a) and $(q_a) = (r_2, s_\alpha)$ and $(q_i) = (r_1, q_a)$.

The inverse problem of Lagrangian mechanics

Problem. When are $\ddot{q}^i = f^i(q, \dot{q})$ equivalent to the Euler-Lagrange equations of a (yet to be determined) **regular Lagrangian** L?

$\rightsquigarrow$ Find multipliers $g_{ij}(q, \dot{q})$ such that $g_{ij}(\ddot{q}^j - f^j) = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}^i}\right) - \dfrac{\partial L}{\partial q^i}$.

Solution. The multipliers must satisfy the **Helmholtz conditions**

$$\det(g_{ij}) \neq 0, \qquad g_{ji} = g_{ij}, \qquad \frac{\partial g_{ij}}{\partial \dot{q}^k} = \frac{\partial g_{ik}}{\partial \dot{q}^j},$$

$$\frac{d}{dt}(g_{ij}) - \nabla^k_j g_{ik} - \nabla^k_i g_{kj} = 0,$$

$$g_{ik}\Phi^k_j = g_{jk}\Phi^k_i,$$

$\nabla^k_j(q, \dot{q}), \Phi^k_j(q, \dot{q})$ made up from derivatives of f^j.

Conversely: if the above has a solution, $\ddot{q}^i = f^i$ is derivable from a Lagrangian.

$\rightsquigarrow$ If a Lagrangian exists, then its Hessian w.r.t. fibre coord. $\dot{q}^i$ is a multiplier!

$\rightsquigarrow$ This is a mixed set of algebraic eq. and PDE for g_{ij} (in $2n$-variables $(q^i, \dot{q}^i)$).

Lagrangians for the first associated second-order system

The first second-order system of interest is of the form

$$\ddot{r}_1 = 0, \quad \ddot{r}_2 = \Gamma_2(r_1)\dot{r}_1\dot{r}_2, \quad \ddot{s}_\alpha = \Gamma_\alpha(r_1)\dot{r}_1\dot{r}_2.$$

Proposition 1. *There is no* **regular** *Lagrangian, i.e.* (g_{ij}) *is always singular.*

The proof follows from the algebraic conditions. ▮

$\rightsquigarrow$ For the nonholonomic particle the system is

$$\ddot{x} = 0, \quad \ddot{y} = -\frac{x}{1+x^2}\dot{x}\dot{y}, \quad \ddot{z} = -\frac{1}{1+x^2}\dot{x}\dot{y}$$

Lagrangians for the second associated 2nd-order system

The second associated system is $\ddot{r}_1 = 0,\ \ddot{q}_a = \Xi_a(r_1)\dot{q}_a\dot{r}_1,$ (no sum over a).

The only non-vanish. comp. of Φ: $\Phi^a_1 = -\frac{1}{2}\dot{r}_1\dot{q}_a(2\Xi'_a - \Xi^2_a),\ \Phi^a_a = \frac{1}{2}\dot{r}^2_1(2\Xi'_a - \Xi^2_a)$.

- Φ-conditions: If $\Phi^a_a \neq 0$, then $\dot{q}_a g_{aa} = -\dot{r}_1 g_{1a}$.
  If $\Phi^a_a \neq \Phi^b_b$ for $a \neq b$, then $g_{ab} = 0$.
- All the other algebraic conditions do not contribute any new information.

1. General case: We only need to determine g_{11} and g_{aa} from the PDE-eq.

It is quite impossible to find the most general solution for (g_{ij}).

$\rightarrow$ In view of the symmetry: assume all g_{ij} to be functions of $(r_1, \dot{r}_1, \dot{q}_a)$ only.

$\rightarrow$ Then: One of the ∇-Helmholtz conditions is $g'_{bb} + \Xi_b(g_{bbb}\dot{q}_b + g_{bb}) = 0$,
where g_{ijk} is $\partial_{\dot{q}^k} g_{ij}$ and $g'_{ij} = \partial_{q^1} g_{ij}$.

$\Rightarrow$ Use an anszatz: Assume that $g_{bbb} = 0$ for all b. Then $g'_{bb} + g_{bb}\Xi_b = 0$.

$\rightarrow$ Therefore: $g_{bb}(r_1, \dot{r}_1) = F_b(\dot{r}_1)\exp(-\xi_b(r_1))$, with $\xi'_b = \Xi_b$ and $F_b(\dot{r}_1)$ still to be determined from the remaning conditions.

After some analysis: $g_{bb} = \dfrac{C_b}{\dot{r}_1}\exp(-\xi_b)$ and $g_{11} = \displaystyle\sum_b \dfrac{C_b \dot{q}_b^2}{\dot{r}_1^3}\exp(-\xi_b) + F_1(\dot{r}_1)$, where $F_1(\dot{r}_1)$ is arbitrary.

⇝ The most general Lagrangian whose Hessian $(g_{ij}) = \left(\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$ is the above multiplier, is (up to a total time derivative):

$$L = \rho(\dot{r}_1) + \tfrac{1}{2}\textstyle\sum_b C_b \exp(-\xi_b)\frac{\dot{q}_b^2}{\dot{r}_1}, \qquad \text{where } d^2\rho/d\dot{r}_1^2 = F_1.$$

⇝ The Lagrangian is regular, as long as $d^2\rho/d\dot{r}_1^2 \neq 0$, and as long as none of the C_b are zero.

2. Special case. Recall:

- The non-vanish. comp. of Φ: $\Phi^a_1 = -\frac{1}{2}\dot r_1 \dot q_a(2\Xi'_a - \Xi^2_a)$, $\Phi^a_a = \frac{1}{2}\dot r_1^2(2\Xi'_a - \Xi^2_a)$.
- Φ-conditions: If $\Phi^a_a \neq 0$, then $\dot q_a g_{aa} = -\dot r_1 g_{1a}$.
  If $\Phi^a_a \neq \Phi^b_b$ for $a \neq b$, then $g_{ab} = 0$.

?? What if one of the assumptions is not valid, say $\Xi_2 = 0$ and thus $\Phi^2_2 = 0$??

$\rightsquigarrow$ Then, there will no longer be a condition that links g_{22} to g_{12}.

$\Rightarrow$ The above Lagrangian is still ok, but the set of all possible Lagr. may be larger.

$\rightsquigarrow$ Apart from $g_{bbb} = 0$, we can take any other ansatz on g_{12} and g_{22}, for example: $g_{12} = 0$.

$\rightsquigarrow$ It can easily be verified that also

$$L = \rho(\dot r_1) + \sigma(\dot r_2) + \frac{1}{2}\left(\sum_\alpha a_\alpha \exp(-\xi_\alpha)\frac{\dot s^2_\alpha}{\dot r_1}\right)$$

is a Lagrangian (for a system with $\Xi_2 = 0$). It is regular as long as both $d^2\rho/d\dot r_1^2$ and $d^2\sigma/d\dot r_2^2$ do not vanish.

Lagrangians for the class

For the nonholonomic systems (in the class), functions ξ_a such that $\xi_a' = \Xi_a$ are:

$$\xi_2 = \ln N \quad \text{and} \quad \xi_\alpha = \ln(NA_\alpha).$$

Proposition 2. *The function*

$$L = \rho(\dot{r}_1) + \frac{1}{2N}\left(C_2\frac{\dot{r}_2^2}{\dot{r}_1} + \sum_\beta C_\beta \frac{\dot{s}_\beta^2}{A_\beta \dot{r}_1}\right),$$

with $d^2\rho/d\dot{r}_1^2 \neq 0$ and all $C_\alpha \neq 0$ is a regular Lagrangian for the second associated systems. If the invariant measure density N is a constant, then also

$$L = \rho(\dot{r}_1) + \sigma(\dot{r}_2) + \frac{1}{2N}\sum_\beta a_\beta \frac{\dot{s}_\beta^2}{A_\beta \dot{r}_1},$$

with $d^2\rho/d\dot{r}_1^2 \neq 0$, $d^2\sigma/d\dot{r}_1^2 \neq 0$ and all $C_\alpha \neq 0$ is a regular Lagrangian for the second associated systems.

The vertically rolling disk in one of those systems with constant invariant measure. The first Lagrangian is:

$$L = \rho(\dot{\varphi}) + \frac{\sqrt{I + mR^2}}{2} \left(C_2 \frac{\dot{\theta}^2}{\dot{\varphi}} + C_3 \frac{\dot{x}^2}{\cos(\varphi)\dot{\varphi}} + C_4 \frac{\dot{y}^2}{\sin(\varphi)\dot{\varphi}} \right)$$

and the second Lagrangian is:

$$L = \rho(\dot{\varphi}) + \sigma(\dot{\theta}) - \frac{\sqrt{I + mR^2}}{2} \left(a_3 \frac{\dot{x}^2}{\cos(\varphi)\dot{\varphi}} + a_4 \frac{\dot{y}^2}{\sin(\varphi)\dot{\varphi}} \right).$$

Hamiltonian formulation and the constraints in phase space

$\rightsquigarrow$ Let us put for convenience $\rho(\dot{r}_1) = \frac{1}{2}I_1\dot{r}_1^2$ and $\sigma(\dot{r}_2) = \frac{1}{2}I_2\dot{r}_2^2$.

$\rightsquigarrow$ Use the Legendre transformation.

Proposition 3. *The Hamiltonian and constraints for the first Lagrangian are given by:*

$$H = \frac{1}{2I_1}\left(p_1 + \tfrac{1}{2}N\left(\frac{p_2^2}{C_2} + \sum_\beta A_\beta \frac{p_\beta^2}{C_\beta}\right)\right)^2, \qquad C_2 p_\alpha = -C_\alpha p_2.$$

In case N is constant, the second Lagrangian gives

$$H = \frac{1}{2I_2}p_2^2 + \frac{1}{2I_1}\left(p_1 + \tfrac{1}{2}N\left(\sum_\beta \frac{A_\beta}{a_\beta}p_\beta^2\right)\right)^2, \quad I_2 N\dot{r}_1 p_\alpha + a_\alpha p_2 = 0,$$

where $\dot{r}_1(r_1, p_1, p_\alpha) = (p_1 + \frac{1}{2}N\sum_\alpha A_\alpha p_\alpha^2/a_\alpha)/I_1$.

For the rolling disk the first Hamiltonian is

$$H = \frac{1}{2J}\left(p_\varphi + \frac{1}{2\sqrt{I+mR^2}}\left(\frac{p_\theta^2}{C_2} - \frac{\cos(\varphi)p_x^2}{C_3} - \frac{\sin(\varphi)p_y^2}{C_4}\right)\right)^2,$$

and the constraints are $C_2 p_x = -C_3 p_\theta$ and $C_2 p_y = -C_4 p_\theta$.

The second Hamiltonian is (with $a_3 = a_4 = -J/\sqrt{I+mR^2}$)

$$H = \frac{1}{2I}p_\theta^2 + \frac{1}{2}\Big(p_\varphi + \tfrac{1}{2}p_x^2\cos(\varphi) + \tfrac{1}{2}p_y^2\sin(\varphi)\Big)^2$$

and the constraints are

$$p_x - p_y = 0, \qquad \dot\varphi p_x - p_\theta = 0,$$

where $\dot\varphi = p_\varphi + \frac{1}{2}\cos(\varphi)p_x^2 + \frac{1}{2}\sin(\varphi)p_y^2$.

Integrators

To compute a **numeric approximation of the solution** of the nonholonomic system we can use either a **nonholonomic integrator** for the **original** Lagrangian and the constraint, or a **variational integrator** for the **new** found Lagrangians.

1. For a **variational integrator** of a system with Lagrangian L, one needs to choose a discrete Lagrangian $L_d(q_1, q_2)$ (a function on $Q \times Q$ which resembles as close as possible the continuous Lagrangian.

$\rightsquigarrow$ A solution $q(t)$ is then discretised by an array q_k which are solutions of the so-called discrete Euler-Lagrange equations

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0.$$

2. On the other hand, for a **nonholonomic integrator** of a nonholonomic system with Lagrangian L and constraints $\omega^a(q)\dot{q}^a$, we need to choose both a discrete Lagrangian L_d and a discrete constraint functions ω_d^a on $Q \times Q$.

$\rightsquigarrow$ The nonholonomic discrete equations are

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = (\lambda_k)_a \omega^a(q_k), \qquad \omega_d^a(q_k, q_{k+1}) = 0.$$

Usually, if Q is a vector space, one takes the discretization in one of the following ways (for certain α and certain h):

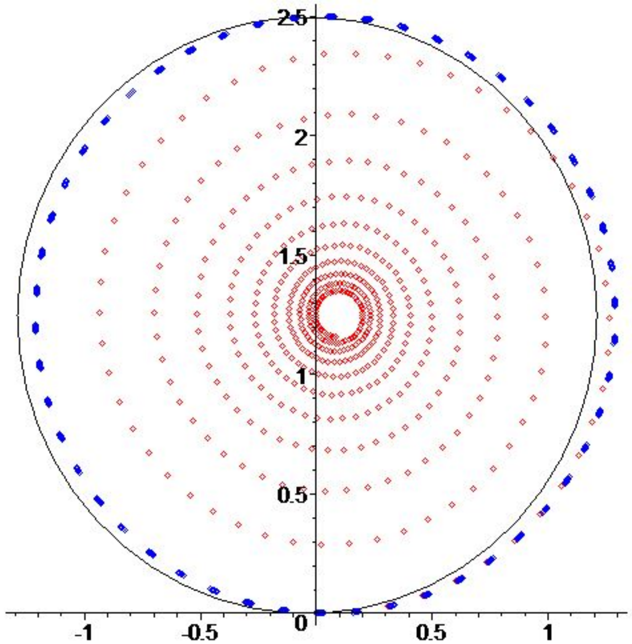
$$L_d(q_1, q_2) = L(q = (1-\alpha)q_1 + \alpha q_2, \dot{q} = \frac{q_2 - q_1}{h})$$

$$\omega_d^a(q_1, q_2) = \omega_i^a(q = (1-\alpha)q_1 + \alpha q_2)\frac{q_2^i - q_1^i}{h}$$

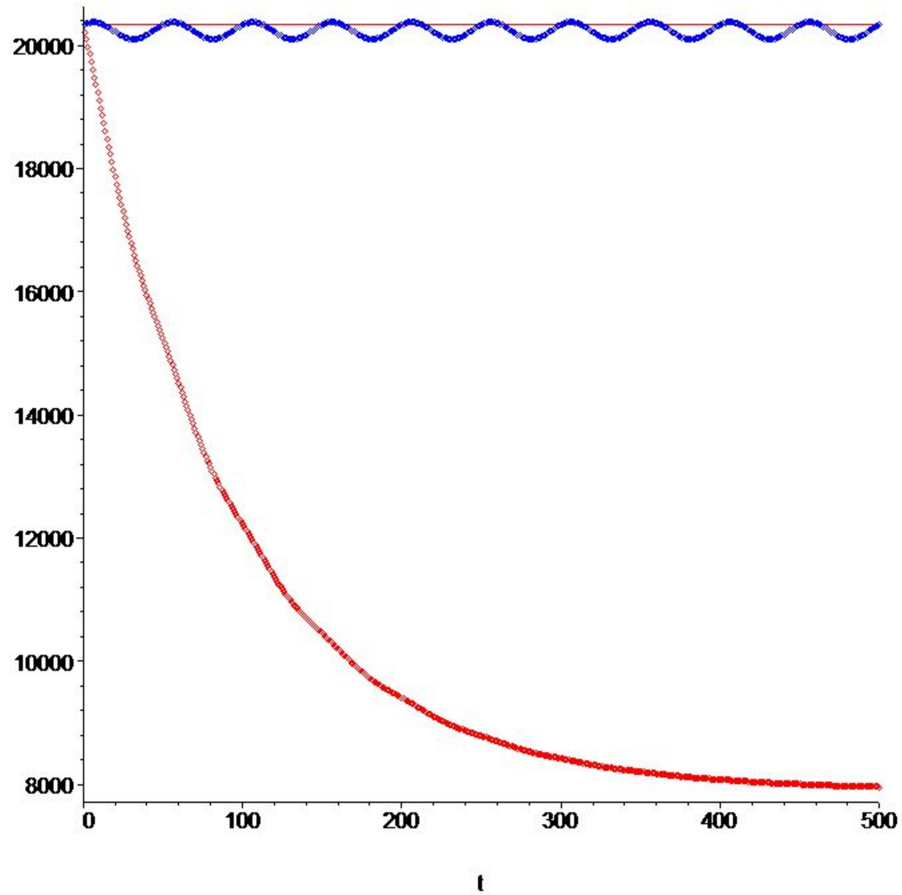
It seems reasonable that if a free Lagrangian for the nonholonomic system exists, the Lagrangian integrator may perform better than a nonholonomic integrator with badly chosen discrete constraints.

The vertically rolling disk

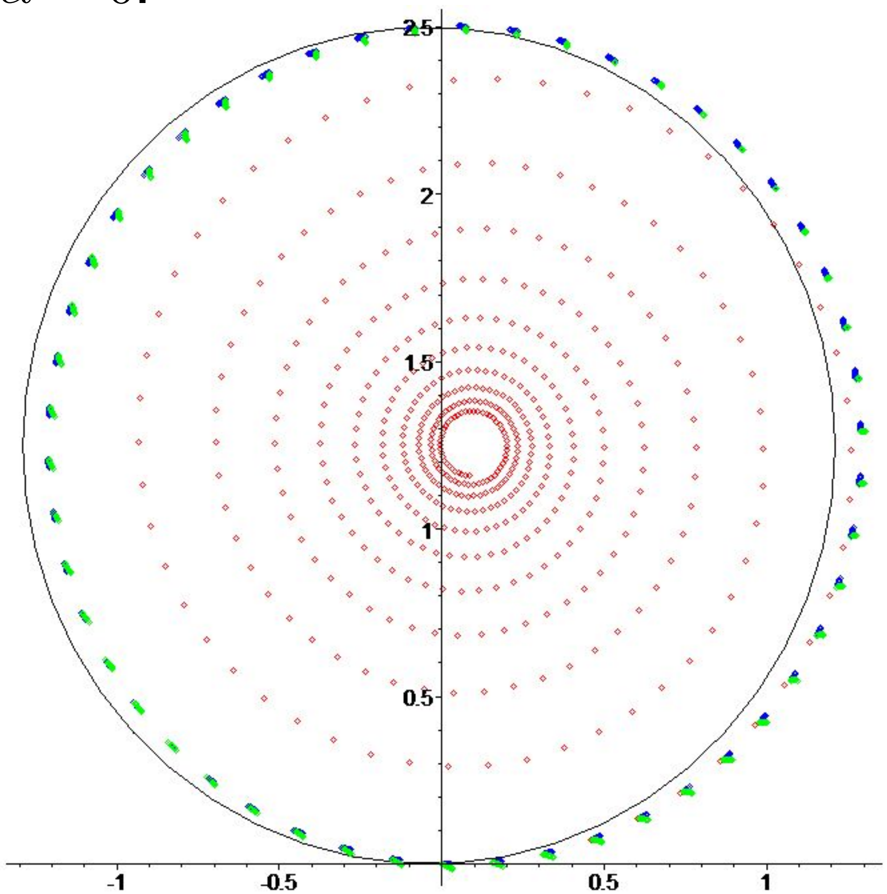
A solution with a **variational integrator** and **nonholonomic integrator** with $\alpha = 0$.



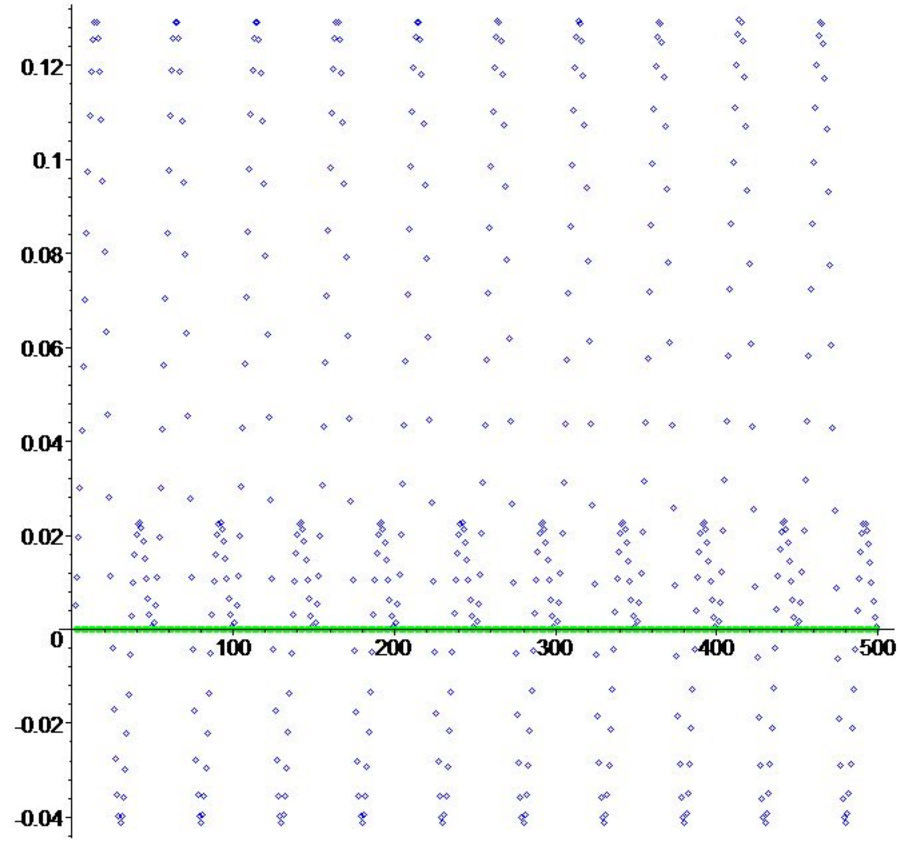
The conservation of energy.



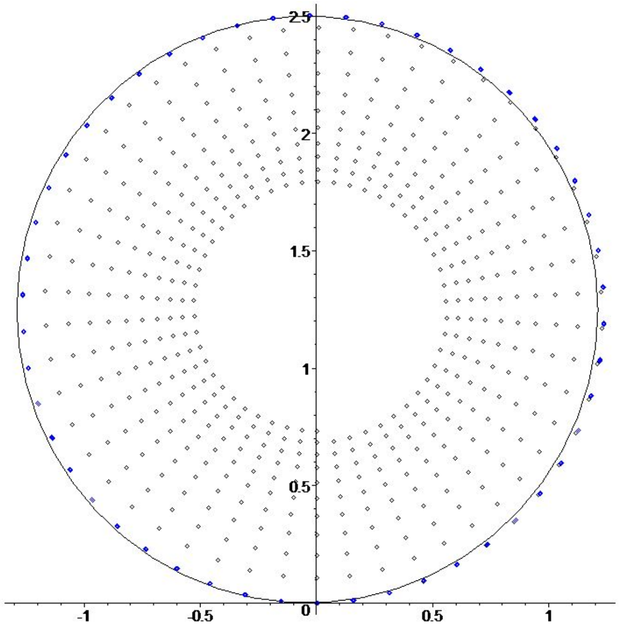
The **'modified' Lagrangian integrator** (considers the constraints as a constant along the nonholonomic motion) with $\alpha = 0$.



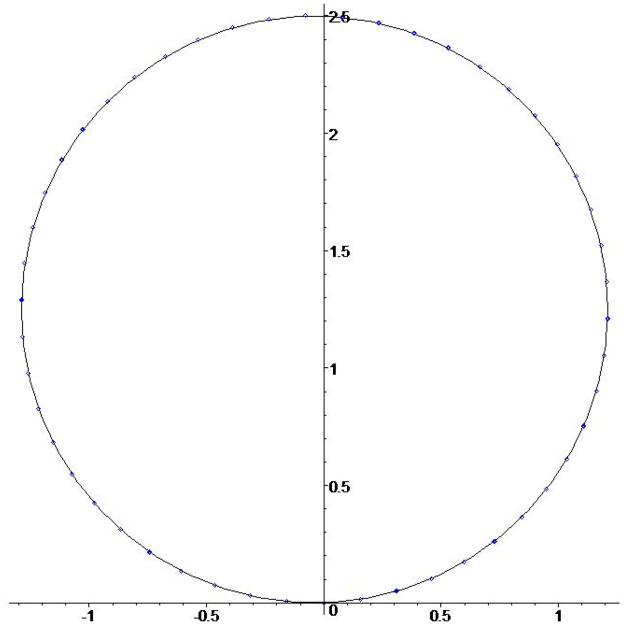
The conservation of the constraints.



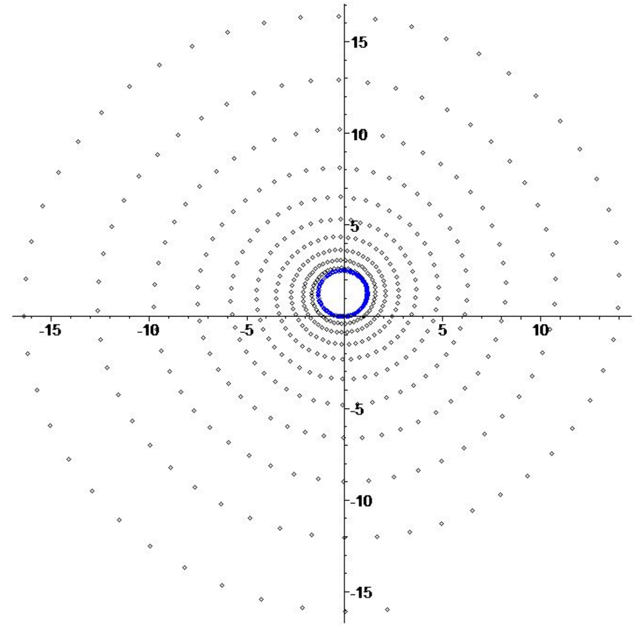
solution with $\alpha = 1/3$



solution with $\alpha = 1/2$

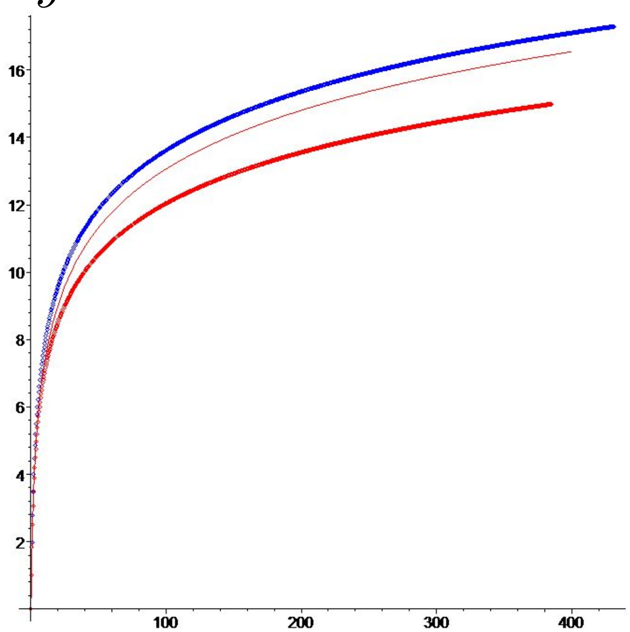


solution with $\alpha = 1$

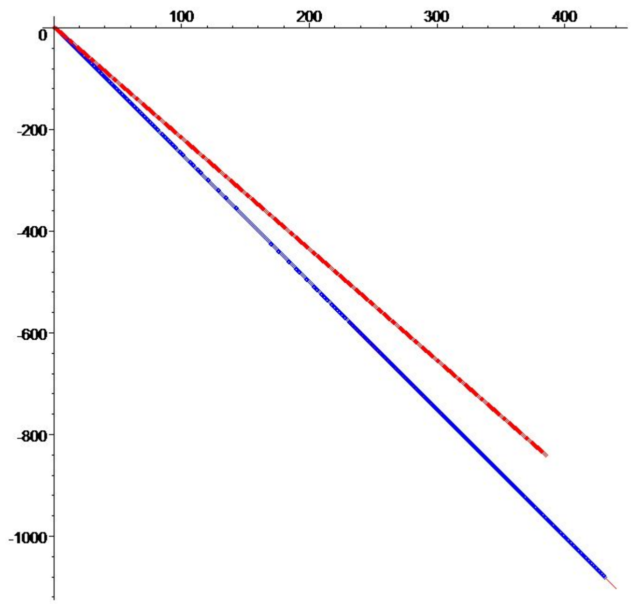


The nonholonomic particle

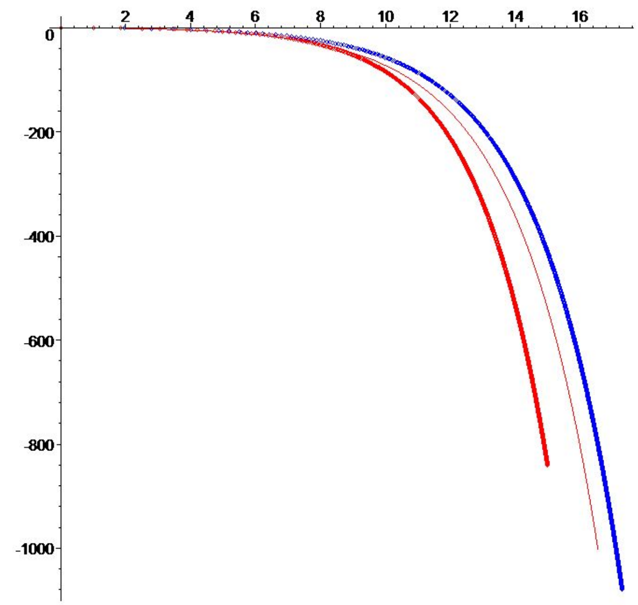
xy-solution with $\alpha = 0$



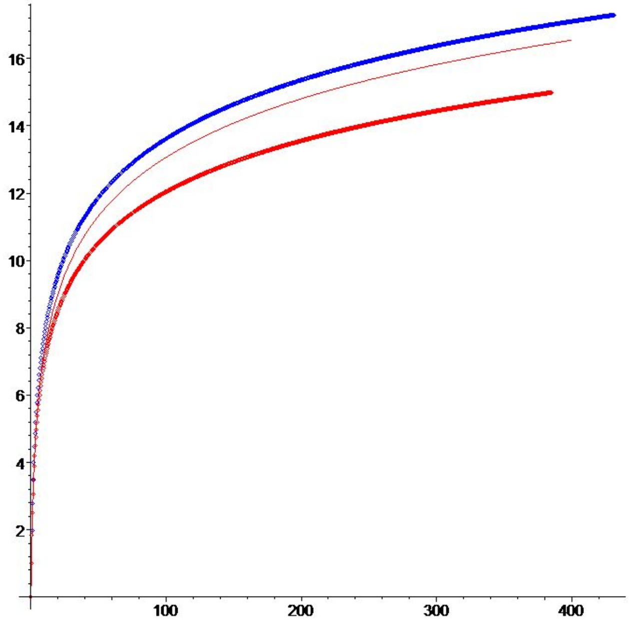
xz-solution with $\alpha = 0$



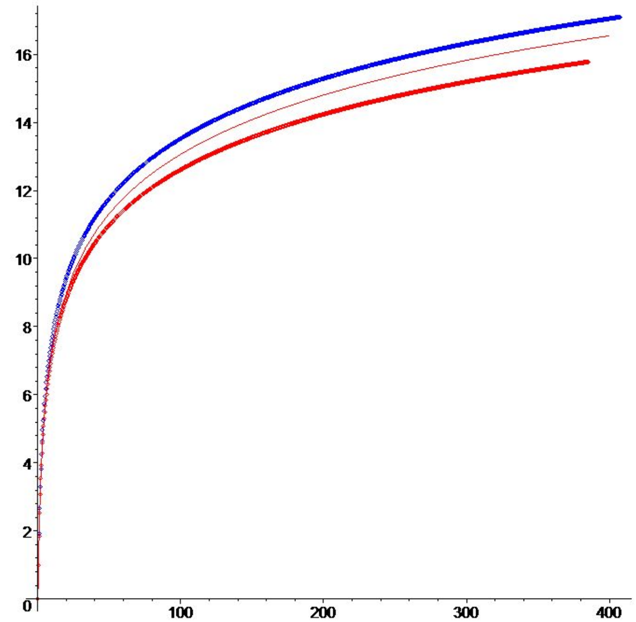
yz-solution with $\alpha = 0$



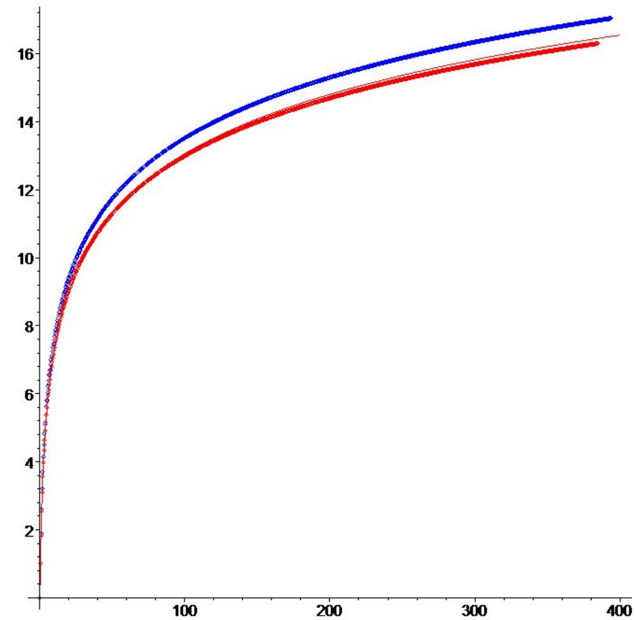
xy-solution with $\alpha = 0$



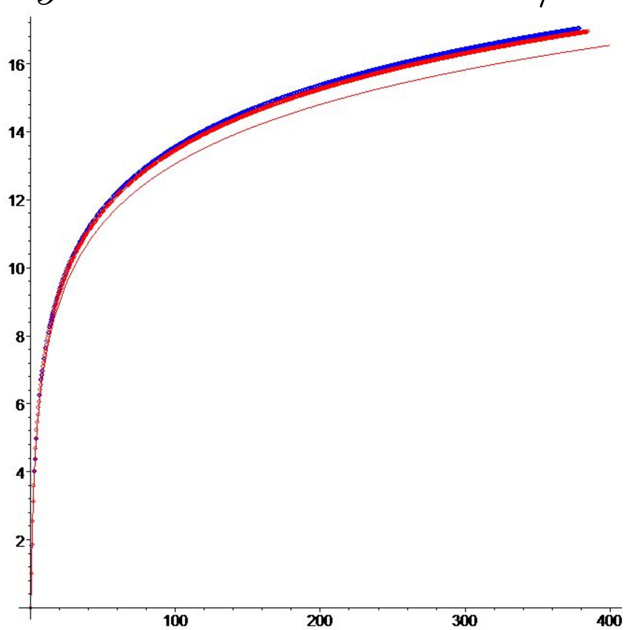
xy-solution with $\alpha = 1/5$



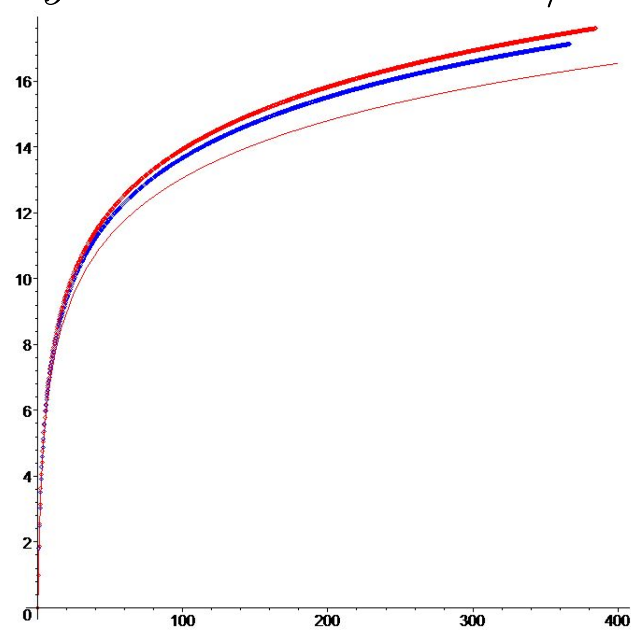
xy-solution with $\alpha = 1/3$



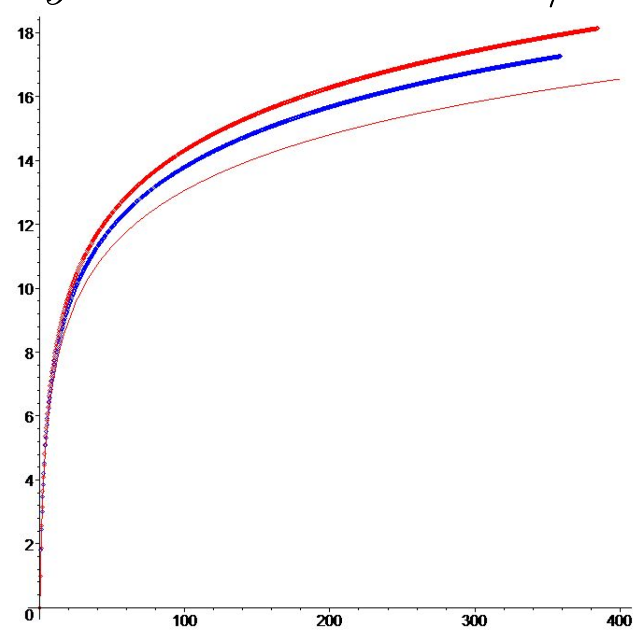
xy-solution with $\alpha = 1/2$



xy-solution with $\alpha = 2/3$



xy-solution with $\alpha = 4/5$



Further systems

What if we add a potential $V(r_2)$?

- Mobile robot with fixed orientation with a potential:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{3}{2}J\dot{\psi}^2 - 10\sin\psi, \qquad \dot{x} = R\cos\theta\dot{\psi}, \dot{y} = R\sin\theta\dot{\psi}.$$

- Knife edge on inclined plane:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\phi}^2 + mgx\sin\alpha, \qquad \dot{x}\sin\phi = \dot{y}\cos\phi.$$

$\rightsquigarrow$ As before, take 'second' associated equations. They are of the form

$$\ddot{r}_1 = 0, \qquad \ddot{r}_2 = \Gamma_2(r_1)\dot{r}_1\dot{r}_2 + t_2(r_1, r_2), \qquad \ddot{s}_\alpha = \Gamma_\alpha(r_1)\dot{r}_1\dot{s}_\alpha + t_\alpha(r_1, r_2).$$

At least in dimension (2+ ($\alpha = 1$)): From the algebraic conditions, there is **no regular Lagrangian** for this associated system.

Choose an other associated system? Which one?

More systems

The Chaplygin sleigh:

$$L = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2 + (I + ma^2)\dot{\theta}^2 - 2ma\sin\theta\dot{x}\dot{\theta} + 2ma\cos\theta\dot{y}\dot{\theta})$$

and $\dot{y}\cos\theta - \dot{x}\sin\theta = 0.$

One kind of associated second-order equations are of the form

$$\begin{aligned}
\ddot{\theta} &= \Gamma_{\theta x\theta}\dot{x}\dot{\theta} + \Gamma_{\theta y\theta}\dot{y}\dot{\theta}, \\
\ddot{x} &= \Gamma_{xx\theta}\dot{x}\dot{\theta} + \Gamma_{x\theta\theta}\dot{\theta}^2 \\
\ddot{y} &= \Gamma_{yy\theta}\dot{y}\dot{\theta} + \Gamma_{y\theta\theta}\dot{\theta}^2
\end{aligned}$$

with $\Gamma_{ijk}(\theta)$. Can we find a Lagrangian for this system?

What about the third system for the vertically rolling disk?

The system was $\ddot{x} = -\dot{y}\dot{\varphi}, \quad \ddot{y} = \dot{x}\dot{\varphi}, \quad \ddot{\varphi} = 0, \quad \ddot{\theta} = 0.$

Does there exist a regular Lagrangian for it? **Yes!** See:

> G. Thompson, Variational connections on Lie groups, *Diff. Geom. Appl.* **18** (2003), 255-270.

A Lagrangian is $L = \frac{1}{2\dot{\varphi}}\Big((\dot{x}^2 - \dot{y}^2)\cos\varphi + 2\dot{x}\dot{y}\sin\varphi\Big) + \rho(\dot{\varphi}) + \sigma(\dot{\theta}),$ where ρ and σ are arbitrary , as long as L they keep regular.

Results in that paper. Any Lie group G has a **canonical connection** ∇,

$$\nabla_X Y = \tfrac{1}{2}[X, Y], \qquad\qquad X, Y \text{ left-invariant VF on } G$$

(The factor $1/2$ makes the torsion of this connection zero).

$\rightsquigarrow$ G. Thompson and coworkers have investigated the IP for the geodesic equations of the associated quadratic spray for (almost all) Lie groups up to dim 6.

$\rightsquigarrow$ In the case of $G = SE(2) \times \mathbf{R}$, we get the above system!

- The system is $\ddot{x} = -\dot{y}\dot{\varphi}, \quad \ddot{y} = \dot{x}\dot{\varphi}, \quad \ddot{\varphi} = 0, \quad \ddot{\theta} = 0.$
- The Lagrangian $L = \frac{1}{2\dot{\varphi}}\Big((\dot{x}^2 - \dot{y}^2)\cos\varphi + 2\dot{x}\dot{y}\sin\varphi\Big) + \rho(\dot{\phi}) + \sigma(\dot{\theta})$.

(Left) multiplication on G induces a action of G on TG:

$$\begin{cases} \psi^G : (a, b, \alpha, \beta) \times (x, y, \varphi, \theta) \mapsto (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \varphi + \alpha, \theta + \\ \psi^{TG} : \qquad \times(\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) \mapsto (\dot{x}\cos\alpha - \dot{y}\sin\alpha, \dot{x}\sin\alpha + \dot{y}\cos\alpha, \dot{\varphi}, \dot{\theta}) \end{cases}$$

$\rightsquigarrow$ The system **is** invariant under this action, but the Lagrangian is **not**!

Problem. Given a (left)-invariant second-order system on a Lie group, when does a (left)-invariant Lagrangian exists

The E-L field in a non-coordinate frame

• We will interpret solutions of $\ddot{q}^i = f^i$ as **integral curves** of the associated **second-order** differential **field** $\Gamma = \dot{q}^i \partial/\partial q^i + f^i \partial/\partial \dot{q}^i$ **on** TQ.

$\rightsquigarrow$ Given a regular **Lagrangian**, the E-L VF Γ_L is completely determined by the assumption that it is a SODE field and by the eq. $\Gamma_L\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$.

• Two important lifts of a VF $Z = Z^i \partial/\partial q^i$ on Q to a VF on TQ:

1. **Complete (tangent) lift** $Z^{\mathrm{C}} = Z^i \frac{\partial}{\partial q^i} + \frac{\partial Z^i}{\partial q^j} \dot{q}^j \frac{\partial}{\partial \dot{q}^i}$.

$\rightsquigarrow$ flow of Z^{C} consists of the tangent maps of the flow of Z.

2. **Vertical lift** $Z^{\mathrm{V}} = Z^i \frac{\partial}{\partial \dot{q}^i}$. $\rightsquigarrow$ tangent to fibres and on T_qQ coincides with Z_q.

• If $\{Z_i\}$ is a **basis of VF on** Q, then an equiv. expression for the E-L eq. is

$$\Gamma_L\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \qquad \Leftrightarrow \qquad \Gamma_L(Z_i^{\mathrm{V}}(L)) - Z_i^{\mathrm{C}}(L) = 0.$$

$\rightsquigarrow$ Similarly: we can rewrite the Helmholtz cond. (on TQ) w.r.t. the basis $\{Z_i\}$.

Vector fields with symmetry

Let $\psi^Q : G \times Q \to Q$ be a (free and proper) **action** of a Lie group G on Q. Then $\pi^Q : Q \to Q/G$ is a principal G-bundle.

1. A vector field Z on Q is **invariant** if $T\psi^Q_g(Z(q)) = Z(\psi^Q_g(q))$.

$\rightsquigarrow$ Z defines a π^Q-related **reduced** vector field $\check{Z}$ on Q/G: the relation

$$T\pi^Q\big(Z(q)\big) = \check{Z}\big(\pi^Q(q)\big)$$

is independent of the choice of $q \in Q$ within $\pi^Q(q) \in Q/G$.

2. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G; $\exp : \mathfrak{g} \to G$ the corresponding exponential map. For $\xi \in \mathfrak{g}$ define the **fundamental** vector field $\tilde{\xi}$ on Q as

$$\tilde{\xi}(q) = \frac{d}{dt}\Big|_{t=0} \left[\psi^Q(\exp(t\xi), q)\right]$$

$\rightsquigarrow$ They are not invariant.

Infinitesimal condition for a VF Z on Q to be **invariant**?

$\rightsquigarrow$ Assuming G is connected: $[\tilde{\xi}, Z] = 0, \forall \xi \in \mathfrak{g}$.

- The second order field Γ is a vector field on TQ.

- The action ψ^Q on Q induces an **action** $\psi_g^{TQ} = T\psi_g^Q$ **on** TQ.

  $\rightsquigarrow$ A function L on TQ is invariant: $L(\psi_g^{TQ}v) = L(v)$.

  $\rightsquigarrow$ A VF Γ on TQ is an **invariant** on TQ if $T\psi_g^{TQ}(\Gamma(v)) = \Gamma(\psi_g^{TQ}(v))$.

- **Infinitesimal condition?** Take $\xi \in \mathfrak{g}$.

$\rightsquigarrow$ The fundamental VF $\tilde{\xi}$ of the action ψ^Q on Q is the infinitesimal generator of the 1-par. group of transformations $\psi^Q_{\exp(t\xi)}$.

$\rightsquigarrow$ The fundamental VFs of the induced action $T\psi_g^Q$ on TQ is the infin. generator of $T\psi^Q_{\exp(t\xi)}$, and is thus $\tilde{\xi}^{\mathrm{C}}$, the **complete lift** of $\tilde{\xi}$!

$\rightsquigarrow$ So: $\tilde{E}_a^{\mathrm{C}}(L) = 0$ and $[\tilde{E}_a^{\mathrm{C}}, \Gamma] = 0$, $\{E_a\}$ basis of $\mathfrak{g}$.

On a Lie group

From now on: The manifold Q will be a **Lie group** G.
The left multiplication L_g induces an action TL_g of G on TG.

$\rightsquigarrow$ **Problem.** When does a **left-invariant** (LI) Lagrangian exist?

Notations and conventions.

1. Let $\{E_i\}$ be a basis of the **Lie algebra** $\mathfrak{g} = T_eG$.

- Let $\{\hat{E}_i\}$ be the corresponding **left-invariant** basis of $\mathcal{X}(G)$.
- Let $\{\tilde{E}_i\}$ be the corresponding **right-invariant** basis of $\mathcal{X}(G)$.

$\rightsquigarrow$ Infinitesimal condition for a VF X on G to be left-invariant: $[\tilde{E}_i, X] = 0$.

$\rightsquigarrow$ Infinitesimal condition for a VF Γ on TG to be left-invariant: $[\tilde{E}_i^{\mathrm{C}}, \Gamma] = 0$.

2. **Fibre coordinates** of $v_g \in T_gG$ w.r.t. $\{\hat{E}_i\}$: $v_g = w^i \hat{E}_i(g)$.

$\rightsquigarrow$ (w^i) are also the coordinates of the element $\xi = TL_{g^{-1}} v_g = w^i E_i$ in the Lie algebra $\mathfrak{g}$.

Invariant systems

Proposition. *If the Lagrangian is left-invariant,* $\tilde{E}_i^{\mathrm{C}}(L) = 0$*, then its Euler-Lagrange field* Γ_L *and its Hessian (multiplier) are left-invariant.*

The converse is **not** true: A left-invariant Γ for which a left-invariant multiplier (a solution of Helmholtz conditions) exists does not automatically induce left-invariant Lagrangians.

Example

Let Γ be the spray of the **canonical connection**, defined as

$$\nabla_X Y = \tfrac{1}{2}[X,Y], \quad X, Y \text{ left-invariant VF on } G.$$

For example: Let G be the **Lie group of the affine line**.

- Elem. of G are of the form $\begin{bmatrix} \exp(q_1) & q_2 \\ 0 & 1 \end{bmatrix}$; those of $\mathfrak{g}$ of the form $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$.
- A basis for $\mathfrak{g}$ consists of $E_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\{E_x, E_y\} = E_y$.
- The basis of right-invariant VF's is $\tilde{E}_x = \dfrac{\partial}{\partial q_1} + q_2 \dfrac{\partial}{\partial q_2}$, $\tilde{E}_y = \dfrac{\partial}{\partial q_2}$ and thus

$$\tilde{E}_x^{\mathrm{C}} = \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + \dot{q}_2 \frac{\partial}{\partial \dot{q}_2}, \quad \tilde{E}_y^{\mathrm{C}} = \frac{\partial}{\partial q_2}.$$

⇝ In this case $\Gamma = \dot{q}_1 \dfrac{\partial}{\partial q_1} + \dot{q}_2 \dfrac{\partial}{\partial q_2} + \dot{q}_1 \dot{q}_2 \dfrac{\partial}{\partial \dot{q}_2}$ and thus $[\tilde{E}_x^{\mathrm{C}}, \Gamma] = 0 = [\tilde{E}_y^{\mathrm{C}}, \Gamma]$.

⇝ Γ is **invariant** (This is in fact true for all Lie groups!).

According to

G. Thompson, Variational connections on Lie groups, *Diff. Geom. Appl.* **18** (2003) 255–270,

the **most general** Lagrangian (subject to the regularity condition), is

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \dot{q}^1 \theta(q_1, q_2, z) + \psi(\dot{q}^1), \qquad z = \dot{q}_2/\dot{q}_1,$$

where ψ is an arbitrary function and θ is a solution of the PDE

$$z\theta_{zz} + z\theta_{zq_2} + \theta_{q_1 z} - \theta_{q_2} = 0 \qquad \text{(subscripts denote derivatives).}$$

⇝ For example, $L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2}\dot{q}_1^2 + \exp(-q_1)\frac{\dot{q}_2^2}{2\dot{q}_1}$ is a Lagrangian.

⇝ **None** of the Lagrangians are **invariant** and regular:

→ If $\tilde{E}_1^{\mathrm{C}}(L) = 0$ and $\tilde{E}_2^{\mathrm{C}}(L) = 0$ then $\theta_{q_1} + z\theta_z = 0$ and $\theta_{q_2} = 0$.

→ Together with the θ-relation: $\theta_z = 0$ and $\theta_{q_1} = 0$.

→ Then, θ is a constant and the Lagrangian is clearly degenerate!

Invariant systems

Proposition. *If the Lagrangian is left-invariant, $\tilde{E}_i^{\rm C}(L)=0$ then its Euler-Lagrange field Γ_L and its Hessian (multiplier) are left-invariant.*

The converse is **not** true: A left-invariant Γ for which a left-invariant multiplier (a solution of Helmholtz conditions) exists does not automatically induce left-invariant Lagrangians.

What is the **obstruction**? Suppose that

(1) Γ is invariant, $[\tilde{E}_i^{\rm C},\Gamma]=0$.

(2) there exists a left-invariant multiplier. If $K_{ij}\in C^\infty(TG)$ are its components w.r.t. the LI basis $\{\hat{E}_i\}$ of VF on G, then $\tilde{E}_k^{\rm C}(K_{ij})=0$.

$\rightsquigarrow$ There is a regular Lagrangian L for Γ, such that $K_{ij}=\hat{E}_i^{\rm V}\hat{E}_j^{\rm V}(L)$.

$\rightsquigarrow$ We have $0=\tilde{E}_k^{\rm C}(K_{ij})=\hat{E}_i^{\rm V}\hat{E}_j^{\rm V}(\tilde{E}_k^{\rm C}(L))$, whence $\tilde{E}_k^{\rm C}(L)=a_{kl}w^l+b_k$, for some functions a_{kl},b_k on G. If a_{kl} and b_k vanish, this Lagrangian is invariant.

What are the **structural properties** of a_{ij} and b_k?

- Since L is known to be a Lagrangian and $\Gamma = w^i \hat{E}_i^{\mathrm{C}} + \Gamma^i \hat{E}_i^{\mathrm{V}}$ is invariant,

$$0 = \tilde{E}_i^{\mathrm{C}} \left(\Gamma(\hat{E}_j^{\mathrm{V}}(L)) - \hat{E}_j^{\mathrm{C}}(L) \right) = \Gamma(a_{ij}) - \hat{E}_j^{\mathrm{C}}(a_{ik} w^k + b_i)$$
$$= w^k \left(\hat{E}_k(a_{ij}) - \hat{E}_j(a_{ik}) - a_{il} C^l_{jk} \right) - \hat{E}_j(b_i).$$

$\rightsquigarrow$ b_i is **constant**

$\rightsquigarrow$ $\hat{E}_k(a_{ij}) - \hat{E}_j(a_{ik}) - a_{il} C^l_{jk} = 0$.Let ϑ^i be the 1-forms on G dual to the $\hat{E}_i$; then $\forall i$, $d(a_{ij}\vartheta^j) = 0$, from which $a_{ij} = \hat{E}_j(f_i)$ for some functions f_i on G.

- Now: $\tilde{E}_i^{\mathrm{C}}(L) = w^j \hat{E}_j(f_i) + b_i$. Structural properties of f_k?

$\rightsquigarrow$ From the identity $0 = \tilde{E}_i^{\mathrm{C}} \tilde{E}_j^{\mathrm{C}}(L) - \tilde{E}_j^{\mathrm{C}} \tilde{E}_i^{\mathrm{C}}(L) + C^k_{ij} \tilde{E}_k^{\mathrm{C}}(L)$

$$= w^k \hat{E}_k \left(\tilde{E}_i(f_j) - \tilde{E}_j(f_i) + C^l_{ij} f_l \right) + C^k_{ij} b_k,$$

we get that $C^k_{ij} b_k = 0$ and that there exist **constants** α_{ij} such that

$$\tilde{E}_i(f_j) - \tilde{E}_j(f_i) + C^l_{ij} f_l = \alpha_{ij}.$$

$\rightsquigarrow$ Let $b : \mathfrak{g} \to \mathbf{R}$, $b(\xi) = b_i \xi^i$. b is a **cocycle** (cohomol. of $\mathfrak{g}$ with values in $\mathbf{R}$).

$\rightsquigarrow$ And α_{ij}? Operate with $\tilde{E}_k$ again on the α-eq and take the cyclic sum. We get $\alpha_{il} C^l_{jk} + \alpha_{jl} C^l_{ki} + \alpha_{kl} C^l_{ij} = 0$. Then $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$, $\alpha(\xi, \eta) = \alpha_{ij} \xi^i \eta^j$ is a **cocycle**.

Recall. "L + total time derivative" gives an other Lagrangian for Γ.

Problem. When can we find an **invariant** Lagrangian among them?

Remark: f_i is determined only up to the addition of a constant ($a_{ij} = \hat{E}_j(f_i)$).
$\rightsquigarrow$ the addition of a constant leaves b unchanged and changes α by a coboundary.

$\rightsquigarrow$ If α_{ij} and b_i are both **cohomologous to zero**, then $b_i = 0$, and by choice of additive constants we can assume that $\tilde{E}_i(f_j) - \tilde{E}_j(f_i) + C^l_{ij} f_l = 0$.

$\rightsquigarrow$ But then $f_i = \tilde{E}_i(f)$ for some function f on G.

$\rightsquigarrow$ But then $L - w^j \hat{E}_j(f) = L - \dot{f}$ is invariant, and of course has Γ as its E-L field and has the same Hessian as L.

Theorem. *An invariant multiplier for an invariant second-order differential equation field Γ determines a cohomology class in $H^1(\mathfrak{g})$ and one in $H^2(\mathfrak{g})$. The field Γ is derivable from an invariant Lagrangian if and only if the corresponding cohomology classes vanish.*

Reduction

Conclusion so far: to solve the invariant inverse problem we need to solve the **Helmholtz** conditions and the above **cohomology** conditions on TG.

$\rightsquigarrow$ even more conditions!! We are dealing with a harder problem!

Not true: we can do a **reduction to the Lie algebra** $\mathfrak{g} = T_eG$: There is a 1-1 corresp. between left-invariant functions F on TG and functions f on $\mathfrak{g}$.

1. F restricted to T_eG is a function f on $\mathfrak{g}$.

2. Given f on $\mathfrak{g}$, define F by $F(v_g) = f(TL_{g^{-1}}v_g)$. Then F is left-invariant: for $TL_h(v_g) \in T_{hg}G$, $F(TL_h(v_g)) = f(TL_{(hg)^{-1}}TL_h(v_g)) = F(v_g)$.

$\rightsquigarrow$ Objects of interest will live on $\mathfrak{g}$: a **vector space**!

Recall: A function F on TG is left-invariant if $\tilde{E}_i^{\mathrm{C}}(F) = 0$.

A second-order field $\Gamma = w^i \hat{E}_j^{\mathrm{C}} + \Gamma^j \hat{E}_j^{\mathrm{V}} \in \mathcal{X}(TG)$ is invariant if $[\tilde{E}_i^{\mathrm{C}}, \Gamma] = 0$, or if $\tilde{E}_i^{\mathrm{C}}(w^j) = 0$ (trivially satisfied) and if $\tilde{E}_i^{\mathrm{C}}(\Gamma^j) = 0$.

$\rightsquigarrow$ Since all Γ^j are left-invariant, they reduce to functions γ^j on $\mathfrak{g}$.

$\rightsquigarrow$ Γ reduces to the vector field $\gamma = \gamma^i \frac{\partial}{\partial w^i} \in \mathcal{X}(\mathfrak{g})$, the **reduced vector field**.

The Euler-Poincaré equations

What is γ for the E-L field Γ_L of a left-invariant Lagrangian $L \in C^\infty(TG)$?

⇝ The E-L equations are $$\begin{aligned} 0 &= \Gamma_L(\hat{E}_i^{\mathrm{V}}(L)) - \hat{E}_i^{\mathrm{C}}(L) \\ &= w^k \hat{E}_k^{\mathrm{C}} \hat{E}_i^{\mathrm{V}}(L) + \Gamma^k \hat{E}_k^{\mathrm{V}} \hat{E}_i^{\mathrm{V}}(L) - \hat{E}_i^{\mathrm{C}}(L). \end{aligned}$$

⇝ Given $\hat{E}_i = A_i^j \tilde{E}_j$, we have $\hat{E}_i^{\mathrm{C}} = A_i^j \tilde{E}_j^{\mathrm{C}} + w^k C_{ki}^j \hat{E}_j^{\mathrm{V}}$ and $\hat{E}_i^{\mathrm{V}} = A_i^j \tilde{E}_j^{\mathrm{V}}$.

⇝ Therefore $\hat{E}_i^{\mathrm{C}}(L) = w^k C_{ki}^j \hat{E}_j^{\mathrm{V}}(L)$ and

$w^k \hat{E}_k^{\mathrm{C}} \hat{E}_i^{\mathrm{V}}(L) = w^k A_k^j \tilde{E}_j^{\mathrm{C}} \hat{E}_i^{\mathrm{V}}(L) = w^k A_k^j \hat{E}_i^{\mathrm{V}} \tilde{E}_j^{\mathrm{C}}(L) + w^k A_k^j [\tilde{E}_j^{\mathrm{C}}, \hat{E}_i^{\mathrm{V}}](L) = 0.$

⇝ The Euler-Lagrange equations are thus $\Gamma^k \hat{E}_k^{\mathrm{V}} \hat{E}_i^{\mathrm{V}}(L) = w^k C_{ki}^j \hat{E}_j^{\mathrm{V}}(L)$.

Let $l \in C^\infty(\mathfrak{g})$ be the restriction of $L \in C^\infty(TG)$ to $\mathfrak{g}$.

⇝ The restriction of $\hat{E}_k^{\mathrm{V}}(L)$ to $\mathfrak{g}$ is $\partial l / \partial w^k$, and so on.

⇝ The defining relation for the **reduced vector field** $\gamma \in \mathcal{X}(\mathfrak{g})$ of Γ is therefore

$$\gamma\left(\frac{\partial l}{\partial w^l}\right) = C_{ml}^j w^m \frac{\partial l}{\partial w^j}.$$

These are the so-called **Euler-Poincaré** equations.

Theorem. *Let* Γ *be an invariant second-order differential equation field on a Lie group* G*, and* γ *the corresponding reduced vector field on* $\mathfrak{g}$*. Then* Γ *admits a regular invariant Lagrangian* L *on* TG *if and only if* γ *admits a regular Lagrangian on* $\mathfrak{g}$*, in the sense that there is a smooth function* l *whose Hessian is non-singular, such that* γ *is the vector field uniquely determined by the Euler-Poincaré equations of* l*.*

The reduced Helmholtz conditions

Recall: The Helmholtz conditions can be cast in a coordinate-free form.

$\rightsquigarrow$ We can compute their expression w.r.t. any basis, say here the LI $\{\hat{E}_i\}$.

$\rightsquigarrow$ The LI components (K_{ij}) on TG define components (k_{ij}) on $\mathfrak{g}$ and vice versa.

$\rightsquigarrow$ We get LI expressions which, when restricted to $\mathfrak{g}$ are conditions for k_{ij}.

In detail:

$$\det(k_{ij}) \neq 0, \qquad k_{ij} = k_{ji}, \qquad \frac{\partial k_{ij}}{\partial w^l} = \frac{\partial k_{lj}}{\partial w^i}.$$

$$\gamma^k \frac{\partial k_{ij}}{\partial w^k} - k_{kj}\lambda^k_i - k_{ik}\lambda^k_j = 0,$$

$$k_{ij}\phi^i_k = k_{ik}\phi^i_j,$$

λ^k_i, ϕ^i_k made up from derivatives of γ^k.

Theorem. *Suppose given an invariant second-order differential equation field Γ, with reduced vector field γ. Then there is an invariant multiplier matrix (K_{ij}) for Γ on TG if and only there is a matrix (k_{ij}) for γ on $\mathfrak{g}$, satisfying the above.*

Then, as before!

Theorem. *A multiplier matrix for* $\gamma \in \mathcal{X}(\mathfrak{g})$ *determines a cohomology class in* $H^1(\mathfrak{g})$ *and one in* $H^2(\mathfrak{g})$. *The vector field* γ *is derivable from a Lagrangian if and only if the corresponding cohomology classes vanish.* ∎

Suppose we have found a multiplier k_{ij} for γ.

$\rightsquigarrow$ From the proof, an expression for the two cocycles (that should be cohomologous to zero for there to be a Lagrangian) is

$$\mu_{ij} = \tfrac{1}{2}\left(\frac{\partial \gamma^l}{\partial w^i} + C^l_{ki} w^k\right) k_{jl} - \tfrac{1}{2}\left(\frac{\partial \gamma^l}{\partial w^j} + C^l_{kj} w^k\right) k_{il}$$

and

$$b_i = \gamma^j(0) k_{ij}(0).$$

Examples

1. The canonical connection.

- If $\nabla_{\hat{E}_i}\hat{E}_j = \Gamma^k_{ij}\hat{E}_k$, the spray is $\Gamma = w^k\hat{E}^{\mathrm{C}}_k + \Gamma^k\hat{E}^{\mathrm{V}}_k$, with $\Gamma^k = \Gamma^k_{ij}w^iw^j$.

- In this case, $\nabla_{\hat{E}_i}\hat{E}_j = \frac{1}{2}C^k_{ij}\hat{E}_k$, so $\Gamma^k = 0$. Therefore $\gamma^k = 0$, the reduced VF γ on $\mathfrak{g}$ is **zero**, and the reduced equations are $\dot{w} = 0$!

Proposition. If there is a left-invariant Lagrangian, then it is also right-invariant. The restriction l to $\mathfrak{g}$ is ad-invariant.

See also: Z. Muzsnay, An invariant variational principle for canonical flows on a Lie group, *J. Math. Phys.* **46** (2005) 112902.

2. Bloch-Iserles equations

See: A. M. Bloch and A. Iserles, On an isospectral Lie-Poisson system and its Lie algebra, *Found. Comput. Math.* **6** (2006) 121-144.

• The space of interest is $\mathrm{Sym}(n)$, the **symmetric $n \times n$ matrices**.

• The equation is $\dot{w} = [w^2, N]$, where $w \in \mathrm{Sym}(n)$ and N is a **skew-symmetric** matrix.

• With the help of N one can give $\mathrm{Sym}(n)$ the structure of a **Lie algebra**, with

$$\{w_1, w_2\} = w_1 N w_2 - w_2 N w_1, \qquad w_1, w_2 \in \mathrm{Sym}(n).$$

$\rightsquigarrow$ Can we find a Lagrangian $l \in C^\infty(\mathrm{Sym}(n))$ for which the above equation is of Euler-Poincaré type with respect to the above Lie algebra?

Consider **the case** $n = 2$.

- For a basis of $\mathrm{Sym}(n)$ we can take

$$E_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and WLOG } N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

- The non-zero brackets: $\{E_x, E_y\} = 2E_z$, $\{E_x, E_z\} = E_y$ and $\{E_y, E_z\} = 2E_z$.

- For $w = xE_x + yE_y + zE_z = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$, the equation $\dot{w} = [w^2, N]$, is

$$\begin{bmatrix} \dot{x} & \dot{y} \\ \dot{y} & \dot{z} \end{bmatrix} = \begin{bmatrix} -2y(x+z) & x^2 - z^2 \\ x^2 - z^2 & 2y(x+z) \end{bmatrix}.$$

$\rightsquigarrow$ Using the techniques described above: the function

$$l(x, y, z) = \tfrac{1}{2}(x^2 + 2y^2 + z^2).$$

is a regular Lagrangian.

3. An illustrative example on the Lie group of the affine line

• As before, a basis for the corresponding Lie algebra is

$$E_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

for which $\{E_x, E_y\} = E_y$.

• Let $A = aE_x + bE_y$ be a constant vector in the Lie algebra. When is the equation

$$\dot{w} = \{w, \{w, A\}\},$$

or, in the above basis,

$$\dot{x} = 0, \qquad \dot{y} = x(bx - ay).$$

of Euler-Poincaré type?

⇝ We were able to decide if an invariant Lagrangian exists or not in all cases of the parameters and to give its most general expression.

Outlook

1. Given an **action** $G \times Q \to Q$ of a Lie group G **on an manifold** Q. Given a sode VF Γ, invariant under this action. Does there exist an invariant Lagrangian?

2. The inverse problem on a Lie algebroid.