Some examples of mechanical systems on algebroids

Paula Balseiro

(in colaboration with Manuel de León
Juan Carlos Marrero y David Martín de Diego)

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1. *Basics definitions*

Definition

An ***algebroid structure*** in the vector bundle $\tau_D : D \to Q$ is a $\mathbb{R}$-linear bracket $B_D : \Gamma(\tau_D) \times \Gamma(\tau_D) \to \Gamma(\tau_D)$ and two vector bundle morphism $\rho^l_D : D \to TQ$, and $\rho^r_D : D \to TQ$ (*anchor maps*), such that

$$B_D(f\sigma, g\bar{\sigma}) = f\rho^l_D(\sigma)(g)\bar{\sigma} - g\rho^r_D(\bar{\sigma})(f)\sigma + fgB_D(\sigma, \bar{\sigma})$$

for $\sigma, \bar{\sigma} \in \Gamma(D)$ and $f, g \in C^\infty(Q)$.

[Grabowski-Urbanski 1999 (Algebroids)], [Grabowska, Grabowski, Urbanski 2006]

- $\tau_D : D \to Q$ an algebroid $(B_D, \rho^l_D, \rho^r_D)$

Suppose that (x^i) are local coordinates on Q and that $\{\sigma_\alpha\}$ is a local basis of the space $\Gamma(\tau_D)$ such that

$$B_D(\sigma_\alpha, \sigma_\beta) = \mathcal{C}^\gamma_{\alpha\beta}\sigma_\gamma$$

$$\rho^l_D(\sigma_\alpha) = (\rho^l_D)^i_\alpha \frac{\partial}{\partial x^i}, \quad \rho^r_D(\sigma_\beta) = (\rho^r_D)^j_\beta \frac{\partial}{\partial x^j}.$$

The local functions $\mathcal{C}^\gamma_{\alpha\beta}, (\rho^l_D)^i_\alpha$ and $(\rho^r_D)^j_\beta$ are called the ***local structure functions*** of the algebroid $\tau_D : D \to Q$.

$\Gamma(\tau_D) : D \to Q$ an algebroid structure $(B_D, \rho_D^l, \rho_D^r)$

$$\Updownarrow$$

Linear tensor Π_{D^*} of type (2,0) on D^*

$$\Downarrow$$

$$\Pi_{D^*} = (\rho_D^r)^i_\alpha \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p_\alpha} - (\rho_D^l)^j_\beta \frac{\partial}{\partial p_\beta} \otimes \frac{\partial}{\partial x^j} - \mathcal{C}^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\alpha} \otimes \frac{\partial}{\partial p_\beta}$$

with (x^i, p_α) the induced local coordinates on D^*.

Linear algebroid structure $\{\ ,\}_{\Pi_{D^*}}$

$$\begin{cases} \{x^i, x^j\}_{\Pi_{D^*}} = 0 \\ \{x^i, p_a\}_{\Pi_{D^*}} = -(\rho_D^l)^i_a \\ \{p_a, x^i\}_{\Pi_{D^*}} = (\rho_D^r)^i_a \\ \{p_a, p_b\}_{\Pi_{D^*}} = -\mathcal{C}^c_{ab} p_c \end{cases}$$

where (x^i, p_a) are the induced coordinates on D^*.

[Grabowski, Urbański(1995, 1999)], [Grabowska, Grabowski, Urbański(2006)].

Definition

A curve $\gamma : I \to D$ is ρ^l_D-*admissible* (respectively, ρ^r_E-*admissible*) if $\frac{d}{dt}(\tau_D \circ \gamma) = \rho^l_D \circ \gamma$ (respectively, $\frac{d}{dt}(\tau_D \circ \gamma) = \rho^r_D \circ \gamma$).

- $H : D^* \to \mathbb{R}$ ***Hamiltonian function*** on D^*.
- ***Hamiltonian vector field*** $\mathcal{H}^{\Pi_{D^*}}_H$ of H with respect to Π_{D^*}, that is,

$$\mathcal{H}^{\Pi_{D^*}}_H(F) = -\{H, F\}_{\Pi_{D^*}}, \quad \text{for } F \in C^\infty(D^*).$$

- The integral curves of the vector field $\mathcal{H}^{\Pi_{D^*}}_H$ are the solutions of the ***Hamilton equations*** for H.

Therefore, the Hamiltonian vector field of H is given by

$$\mathcal{H}_H^{\Pi_{D^*}} = (\rho_D^l)_\alpha^i \frac{\partial H}{\partial p_\alpha} \frac{\partial}{\partial x^i} - \left((\rho_D^r)_\beta^j \frac{\partial H}{\partial x^j} - \mathcal{C}_{\alpha\beta}^\gamma p_\gamma \frac{\partial H}{\partial p_\alpha} \right) \frac{\partial}{\partial p_\beta},$$

which implies that the local expression of the Hamilton equations is

$$\begin{aligned} \frac{dx^i}{dt} &= (\rho_D^l)_\alpha^i \frac{\partial H}{\partial p_\alpha}, \\ \frac{dp_\beta}{dt} &= -\left\{ (\rho_D^r)_\beta^j \frac{\partial H}{\partial x^j} - \mathcal{C}_{\alpha\beta}^\gamma p_\gamma \frac{\partial H}{\partial p_\alpha} \right\}. \end{aligned}$$

1. *Basic definitions*
2. *Examples:*
    1. *Symmetric Case: Gradient extensions.*

1. *Basic definitions*
2. ***Examples:***
    1. ***Symmetric Case: Gradient extensions.***

Let Q be an n-dimensional manifold. Let $\mathcal{G}$ a riemannian metric on Q. For $X \in \mathfrak{X}(Q)$

$$\dot{x} = X(x)$$

- $\mathcal{G} = \mathcal{G}_{ij}(x) dx^i \otimes dx^j$.
- $\flat_{\mathcal{G}} : \mathfrak{X}(Q) \longrightarrow \Lambda^1(Q), \qquad \sharp_{\mathcal{G}} : \Lambda^1(Q) \longrightarrow \mathfrak{X}(Q)$
- $f \in C^\infty(Q)$, $\text{grad}_{\mathcal{G}}(f) = \sharp_{\mathcal{G}}(df) \quad \Rightarrow \text{grad}_{\mathcal{G}}(f) = \mathcal{G}^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}$ where $(\mathcal{G}^{ij})$ is the inverse matrix of $(\mathcal{G}_{ij})$.
- ***Levi-Civita connection*** $\overset{\mathcal{G}}{\nabla}$

$$[X, Y] = \overset{\mathcal{G}}{\nabla}_X Y - \overset{\mathcal{G}}{\nabla}_Y X \text{ (torsion-free)}$$
$$X(\mathcal{G}(Y, Z)) = \mathcal{G}(\overset{\mathcal{G}}{\nabla}_X Y, Z) + \mathcal{G}(Y, \overset{\mathcal{G}}{\nabla}_X Z) \text{ (metricity)},$$

where $X, Y, Z \in \mathfrak{X}(Q)$.

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where $X, Y, Z \in \mathfrak{X}(Q)$.

- ***Symmetric product:***

$$B_{TQ}(X,Y) = \overset{\mathcal{G}}{\nabla}_X Y + \overset{\mathcal{G}}{\nabla}_Y X \qquad X, Y \in \mathfrak{X}(Q).$$

Locally, $B_{TQ}(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = \left(\Gamma^i_{jk} + \Gamma^i_{kj}\right) \frac{\partial}{\partial x^i} = 2\Gamma^i_{jk} \frac{\partial}{\partial x^i}.$

- ***The anchor maps:***

$$\rho^l_{TQ} = \mathrm{id}_{TQ} \quad \text{and} \quad \rho^r_{TQ} = -\mathrm{id}_{TQ}$$

The tangent bundle equipped with $(B_{TQ}, \mathrm{id}_{TQ}, -\mathrm{id}_{TQ})$ is an (symmetric) algebroid.

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$$\boxed{\begin{array}{c}\text{An algebroid } (B_{TQ}, \mathrm{id}_{TQ}, -\mathrm{id}_{TQ}) \\ \Updownarrow \\ \text{Linear bivector } \Pi_{T^*Q} \text{ in } T^*Q\end{array}}$$

$$\Pi_{T^*Q} = -\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p_j} - \frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial x^i} - 2\Gamma^k_{ij} p_c \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial p_j}$$

where (x^i, p_a) are the induced coordinates on T^*Q.

Symmetric structure $\{\,,\}_{\Pi_{D^*}}$
$$\left\{\begin{array}{l}\{x^i, x^j\}_{\Pi_{T^*Q}} = 0 \\ \{x^i, p_j\}_{\Pi_{T^*Q}} = -1 \\ \{p_i, x_j\}_{\Pi_{T^*Q}} = -1 \\ \{p_i, p_j\}_{\Pi_{T^*Q}} = -2p_k\Gamma^k_{ij}\end{array}\right.$$

For $X \in \mathfrak{X}(Q)$

- ***The hamiltonian function*** $H_X : T^*Q \longrightarrow \mathbb{R}$

  $$H_X(\kappa_x) = \langle \kappa_x, X_x \rangle \qquad \text{that is,} \quad H_X(x^i, p_i) = p_i X^i(x).$$

- $(T^*Q, \{\ ,\}_{\Pi_{T^*Q}}, \mathrm{id}_{TQ}, -\mathrm{id}_{TQ}, h)$ is a hamiltonian system,
- ***The Hamiltonian vector field*** $\mathcal{H}_H^{\Pi_{T^*Q}}$ on T^*Q:

  $$\mathcal{H}_{H_X}^{\Pi_{T^*Q}}(F) = -\{H_X, F\}_{\Pi_{T^*Q}}, \ \text{ for } F \in C^\infty(T^*Q).$$

- ***Hamilton equations:***

  $$\begin{aligned} \dot{x}^i &= X^i(x) \\ \dot{p}_j &= p_k \left(\frac{\partial X^k}{\partial x^j} + 2\Gamma_{ij}^k X^i \right) \end{aligned}$$

These equations are the *gradient extension* of the nonlinear equation $\dot{x}^i = X^i(x)$.

[Cortés,van der Shaft,Crouch(2005)]

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    2. ***Skew-symmetric Case: Nonholonomic Mechanics.***

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Nonholonomic mechanical systems

- $\tau_E : E \to Q$ a Lie algebroid with structure $(\llbracket \cdot\,, \cdot \rrbracket, \rho_E)$.
- $\mathcal{G} : E \times_Q E \to \mathbb{R}$ be a fiber metric.

The systems considered are the *nonholonomic mechanical systems* determined by:

- A Lagrangian function L:

  $$L(e) = \frac{1}{2}\mathcal{G}(e, e) - V(\tau(e)), \quad e \in E,$$

  with V a potential function on Q.
- Nonholonomic (linear) constraints determined by a vector subbundle D of E.

Let us consider the orthogonal decomposition $E = D \oplus D^{\perp}$, and the associated orthogonal projectors

$$P : E \longrightarrow D \qquad Q : E \longrightarrow D^{\perp}$$

We choose a local basis of sections of E, $\{\sigma_{\alpha}\}$, adapted to (L, D), that is:

(i) $\{\sigma_{\alpha}\}$ is an orthogonal basis with respect to $\mathcal{G}$ (i.e., $\mathcal{G}(\sigma_{\alpha}, \sigma_{\beta}) = \delta_{\alpha\beta}$)

(ii) $\{\sigma_{\alpha}\} = \{\sigma_{a}, \sigma_{A}\}$ where $D = \operatorname{span}\{\sigma_{a}\}$, $D^{\perp} = \operatorname{span}\{\sigma_{A}\}$,

(iii) and denote $(x^{i}, y^{\alpha}) = (x^{i}, y^{a}, y^{A})$ the induced coordinates on E

Motion equations of the nonholonomic mechanical system

$\gamma(t) = (x^i(t), y^\alpha(t))$ a curve in E

$$\gamma \text{ is a solution}$$

$$\Updownarrow$$

$$\dot{x}^i = (\rho_E)^i_\alpha y^\alpha$$

$$\overset{\mathcal{G}}{\nabla}_{\gamma(t)} \gamma(t) - \mathrm{grad}_{\mathcal{G}} V(\gamma(t)) \in D^\perp_{\gamma(t)}$$

$$y^A = 0.$$

or equivalently

$$\dot{x}^i = (\rho_E)^i_a y^a$$

$$\dot{y}^a + \mathcal{C}^c_{ab} y^b y^c + (\rho_E)^i_a \frac{\partial V}{\partial x^i} = 0$$

Nonholonomic mechanics & Mechanics in skew-symmetric algebroids

$\tau_D : D \to Q$ ***has a skew-symmetric algebroid structure:***

For all $\sigma, \bar{\sigma} \in \Gamma(\tau_D)$ we define

- $B_D(\sigma, \bar{\sigma}) = P\left(\llbracket i_D \circ \sigma, i_D \circ \bar{\sigma} \rrbracket\right)$
- $\rho_D(\sigma) = \rho_E(i_D \circ \bar{\sigma})$

Then the local structure functions are

- $B_D(\sigma_a, \sigma_b) = P\left(\llbracket \sigma_a, \sigma_b \rrbracket\right) = P\left(\mathcal{C}^c_{ab}\sigma_c + \mathcal{C}^\alpha_{ab}\sigma_\alpha\right) = \mathcal{C}^c_{ab}\sigma_c,$
- $\rho_D(\sigma_a) = (\rho_E)^i_a \frac{\partial}{\partial x^i}$

$$\begin{tikzcd}
D \arrow[rr, hook, "\rho_D"] \arrow[dr, "\tau_D"'] & & TQ \arrow[dl, "\tau"] \\
& Q &
\end{tikzcd}$$

$$\boxed{\begin{array}{c}\text{A skew-symmetric algebroid } (D, B_D, \rho_D) \\ \Updownarrow \\ \text{Linear bivector } \Pi_{D^*} \text{ in } D^*\end{array}}$$

$$\Pi_{D^*} = (\rho_D)^i_a \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_a} - \frac{1}{2}\mathcal{C}^c_{ab} p_c \frac{\partial}{\partial p_a} \wedge \frac{\partial}{\partial p_b}.$$

where (x^i, p_a) are the induced coordinates on D^*.

Linear almost Poisson structure $\{\ ,\}_{\Pi_{D^*}}$

$$\begin{cases} \{x^i, x^j\}_{\Pi_{D^*}} = 0 \\ \{x^i, p_a\}_{\Pi_{D^*}} = (\rho_D)^i_a \\ \{p_a, p_b\}_{\Pi_{D^*}} = -\mathcal{C}^c_{ab} p_c \end{cases}$$

- Let $h : D^* \to \mathbb{R}$ be a ***Hamiltonian function*** on D^*.
- $(D^*, \{\ ,\ \}_{\Pi_{D^*}}, \rho_D, h)$ is a hamiltonian system,
- ***The Hamiltonian vector field*** $\mathcal{H}_h^{\Pi_{D^*}}$ on D^*:

$$\mathcal{H}_h^{\Pi_{D^*}}(F) = -\{h, F\}_{\Pi_{D^*}}, \quad \text{for } F \in C^\infty(D^*).$$

- ***Hamilton equations:***

$$\begin{aligned} \dot{x}^i &= (\rho_D)^i_a \frac{\partial h}{\partial p_a} \\ \dot{p}_a &= -\left((\rho_D)^i_a \frac{\partial h}{\partial x^i} + \mathcal{C}^c_{ab} p_c \frac{\partial h}{\partial p_b} \right) \end{aligned}$$

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    3. ***Mixed Cases:***
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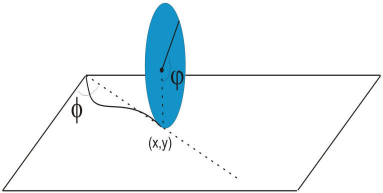
Motivation



The rolling disk

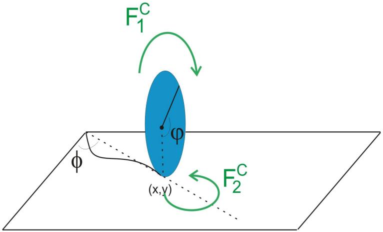
- Configuration manifold: $Q = R^2 \times S^1 \times S^1$
- The lagrangian:

$$L(x, y, \theta, \varphi) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} J\dot{\varphi}^2$$

- The restrictions:

$$\left\{ \begin{array}{lcl} \dot{x} & = & R \cos\varphi \, \dot{\theta} \\ \dot{y} & = & R \, sen \varphi \, \dot{\theta} \end{array} \right.$$

Motivation



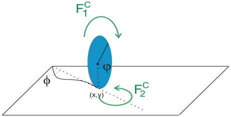
Control Forces: $F^c = \lambda(d\phi - d\theta) \implies \dot{\theta} = -\dot{\phi}.$

- ***Kinematic constraints:***

$$D = \mathrm{span}\{r\cos\theta\frac{\partial}{\partial x} + r\sin\theta\frac{\partial}{\partial y} - \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\}$$

- ***Variational constraints:***

$$\tilde{D} = \mathrm{span}\{r\cos\theta\frac{\partial}{\partial x} + r\sin\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\}$$



$$\gamma(t) = (q^i(t), v^i(t)) = \Big((x^i(t), y^i(t), \theta^i(t), \varphi^i(t)); (\dot{x}^i(t), \dot{y}^i(t), \dot{\theta}^i(t), \dot{\varphi}^i(t))\Big)$$

$\gamma(t)$ in E is a solution of the *generalized nonholonomic system*

$$\Updownarrow$$

$$\gamma(t) \in D_{q(t)}$$
$$\overset{\mathcal{G}}{\nabla}_{\gamma(t)} \gamma(t) - \mathrm{grad}_{\mathcal{G}} V(\gamma(t)) = F^c(\gamma(t)).$$

$$\Updownarrow \quad + \, T_qQ = D_q \oplus \tilde{D}_q^{\perp}$$

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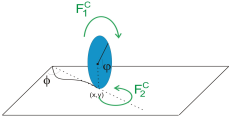

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Geometric formulation of the generalized nonholonomic mechanics

- $\tau_E : E \to Q$ a Lie algebroid $(\llbracket \cdot, \cdot \rrbracket, \rho_E)$.
- Let $\mathcal{G} : E \times_Q E \to \mathbb{R}$ a fiber metric.

A *generalized nonholonomic mechanical system* is determined by:

- A lagrangian function L:

$$L(e) = \frac{1}{2}\mathcal{G}(e, e) - V(\tau(e)), \quad e \in E,$$

  with V a potential function on M.
- **Kinematic constraints** determined by the distribution D of E,
- **Variational constraints** determined by the distribution $\widetilde{D}$.

We assume the *compatibility condition* $E = D \oplus \widetilde{D}^{\perp}$.

Given local coordinates (x^i) in Q and a local basis of sections in E, $\{\sigma_\alpha\}$, adapted to the generalized nonholonomic problem $(L, D, \widetilde{D})$:

(i) $\{\sigma_\alpha\} = \{\sigma_a, \sigma_A\}$ where $D = \mathrm{span}\{\sigma_a\}$, $\widetilde{D}^{\perp} = \mathrm{span}\{\sigma_A\}$.

(ii) $\{\sigma_a\}$ and $\{\sigma_A\}$ are orthogonal with respect to $\mathcal{G}$.

(iii) and denote $(x^i, y^\alpha) = (x^i, y^a, y^A)$ the induced coordinates on E.

Motion equations of the generalized nonholonomic system

$\gamma(t) = (x^i(t), y^\alpha(t))$ curve in E

γ is a solution

$$\Updownarrow$$

$$\begin{cases} \dot{x}^i = (\rho_E)^i_\alpha y^\alpha \\ \overset{\mathcal{G}}{\nabla}_{\gamma(t)}\, \gamma(t) + \mathrm{grad}_{\mathcal{G}} V(\gamma(t)) \in \tilde{D}^{\perp}_{\tau_D(\gamma(t))} \\ y^A = 0 \end{cases}$$

After some computations we get

$$\begin{aligned}
\dot{y}^c &= \mathcal{G}^{cd}\Big[\mathcal{C}^a_{bd} + \mathcal{G}_{aB}\mathcal{C}^B_{bd} - \mathcal{G}_{dA}\mathcal{C}^a_{bA} - \mathcal{G}_{dA}\mathcal{G}_{aB}\mathcal{C}^B_{bA} \\
&\quad + \mathcal{G}_{dA}\frac{\partial \mathcal{G}_{aA}}{\partial x^i}(\rho_E)^i_b\Big]y^a y^b \\
&\quad - \mathcal{G}^{cd}\left((\rho_E)^i_d - \mathcal{G}_{dA}(\rho_E)^i_A\right)\frac{\partial V}{\partial x^i} \\
\dot{x}^i &= (\rho_E)^i_a y^a
\end{aligned}$$

where $\mathcal{G}^{\alpha\beta} = [\mathcal{G}^{-1}]_{\alpha\beta}$

Then we rewrite the motion equations as:

$$\begin{aligned}
\dot{y}^c &= y^a y^b \widetilde{\mathcal{C}}^a_{bc} - (\rho^r_D)^i_c \frac{\partial V}{\partial x^i} \\
\dot{x}^i &= (\rho^l_D)^i_a y^a.
\end{aligned}$$

For all $\sigma, \bar{\sigma} \in \Gamma(\tau_D)$ we define

- $B_D(\sigma, \bar{\sigma}) = P\left(\llbracket i_D \circ \sigma, i_{\widetilde{D}} \circ \Pi\bar{\sigma} \rrbracket\right)$
- $\rho^l_D = \rho_E \circ i_D$
- $\rho^r_D = \rho_E \circ i_{\widetilde{D}} \circ \Pi$

where

$$P : D \oplus D^\perp \longrightarrow D \;\text{ and }\; \Pi : \widetilde{D} \oplus D^\perp \longrightarrow \widetilde{D}$$

and

$$\begin{aligned} B_D(\sigma_a, \sigma_b) &= \widetilde{\mathcal{C}}^c_{ab}\sigma_c, \\ \rho^l_D(\sigma_a) &= (\rho_E)^i_a \frac{\partial}{\partial x^i} \\ \rho^r_D(\sigma_a) &= \mathcal{G}^{ad}((\rho_E)^i_d - \mathcal{G}_{dA}(\rho_E)^i_A)\frac{\partial}{\partial x^i}. \end{aligned}$$

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\end{aligned}$$

$$\boxed{\begin{array}{c}\text{An algebroid } (D, B_D, \rho^l_D, \rho^r_D) \\ \Updownarrow \\ \text{2-contravariant linear tensor } \Pi_{D^*} \text{ in } D^* \end{array}}$$

$$\Pi_{D^*} = (\rho^l_D)^i_a \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial p_a} - (\rho^r_D)^i_a \frac{\partial}{\partial p_a} \otimes \frac{\partial}{\partial x^i} - \widetilde{\mathcal{C}}^c_{ab} p_c \frac{\partial}{\partial p_a} \otimes \frac{\partial}{\partial p_b}$$

where (x^i, p_a) are the induced coordinates on D^*.

Linear structure $\{\ ,\ \}_{\Pi_{D^*}}$

$$\left\{ \begin{array}{l} \{x^i, x^j\}_{\Pi_{D^*}} = 0 \\ \{x^i, p_a\}_{\Pi_{D^*}} = -(\rho^l_D)^i_a \\ \{p_a, x^i\}_{\Pi_{D^*}} = (\rho^r_D)^i_a \\ \{p_a, p_b\}_{\Pi_{D^*}} = -\widetilde{\mathcal{C}}^c_{ab} p_c \end{array} \right.$$

- $h : D^* \to R$ a hamiltonian function on D^*.
- $(D, \{\ ,\ \}_{\Pi_{D^*}}, \rho^l_D, \rho^r_D, h)$ a hamiltonian system,
- ***The Hamiltonian vector field*** $\mathcal{H}^{\Pi_{D^*}}_h$ on D^*:

$$\mathcal{H}^{\Pi_{D^*}}_h(F) = -\{h, F\}_{\Pi_{D^*}}, \quad \text{for } F \in C^\infty(D^*).$$

- ***Hamilton equations:***

$$\begin{aligned} \dot{x}^i &= (\rho^l_D)^i_a \frac{\partial h}{\partial p_a} \\ \dot{p}_b &= -\left((\rho^r_D)^i_a \frac{\partial h}{\partial x^i} - \widetilde{\mathcal{C}}^c_{ab} p_c \frac{\partial h}{\partial p_a} \right) \end{aligned}$$

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2. ***Examples:***
    1. *Symmetric Case: Gradient extensions.*
    2. *Skew-symmetric Case: Nonholonomic Mechanics.*
    3. *Mixed Case:*
        1. *Generalized nonholonomic Mechanics.*
        2. *Nonholonomic systems with dissipation (affgebroids)*

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Nonholonomic systems with friction

- $\tau_V : V \to Q$ a skew-symmetric algebroid structure $((\mathcal{C}_V)^c_{ab}, (\rho_V)^i_a)$
- Equations of motion, in coordinates (x^i, y^a) of V.:

$$\begin{aligned} \dot{x}^i &= (\rho_V)^i_a y^a \\ \dot{y}^b &= y^a (\mathcal{C}_V)^c_{ab} y^c + (\rho_V)^i_b \frac{\partial V}{\partial x^i}. \end{aligned}$$

- To model ***viscous friction***, we define a ***Rayleigh's dissipation function*** as

$$R(q,y) = \frac{1}{2} c_b (\beta^b(q).y)^2$$

$$\Downarrow$$

forcing function:

$$\tau_b(q,y) = -\frac{\partial R}{\partial y^b} = -c_d \beta^d_a(q) \beta^c_b(q) y^a.$$

Rayleigh's dissipation function: $\tau(q,y).y < 0$.

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- Modified equations of motion

$$\begin{aligned}\dot{x}^i &= (\rho_V)^i_a y^a \\ \dot{y}^b &= y^a (\mathcal{C}_V)^c_{ab} y^c + (\rho_V)^i_b \frac{\partial V}{\partial x^i} + \tau_b(q, y)\end{aligned}$$

In coordinates (x^i, p_a) on V^*:

$$\begin{aligned}\dot{x}^i &= (\rho_V)^i_a p_a \\ \dot{p}_b &= p_a[(\mathcal{C}_V)^c_{ab} p_c - c_d \beta^d_a(q) \beta^c_b(q)] + (\rho_V)^i_b \frac{\partial V}{\partial x^i}.\end{aligned}$$

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Now, our aim is to endow the vector bundle $\tau_{\mathcal{A}} : \mathcal{A} \to Q$ with an affine structure.

Affine Structures

Let $\tau_{\mathcal{A}} : \mathcal{A} \to Q$ an affine vector bundle with associated vector bundle $\tau_V : V \to Q$.

The affine dual bundle $\tau_{\mathcal{A}^+} : \mathcal{A}^+ \to Q$

$$\mathcal{A}^+ = \{\varphi_x : \mathcal{A}_x \to R \text{ such that } \varphi \text{ is affine}\}$$

- *distinguished section*

$$1_{\mathcal{A}} : Q \to \mathcal{A}^+,\ 1_{\mathcal{A}}(x)(a) = 1 \ \ \forall x \in Q,\ \forall a \in \mathcal{A}$$

- $\varphi_x = \varphi_x^l + \lambda_x 1_{\mathcal{A}}$ with $\varphi_x^l \in V_x^*$.
- $\mu_{\mathcal{A}^+} : \mathcal{A}^+ \to V^*$ such that $\mu_{\mathcal{A}^+}(\varphi) = \varphi^l$.

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The bidual bundle $\tilde{\mathcal{A}} \to M$

$\tilde{\mathcal{A}} = (\mathcal{A}^+)^*$

- $\tilde{\mathcal{A}}$ is a skew-symmetric algebroid and $\dim \tilde{\mathcal{A}} = \dim V + 1$
- $\{\sigma_a, \sigma_0\}$ a local basis of $\tilde{\mathcal{A}}$
- (x^i, y_a, y_0) local coordinates of $\tilde{\mathcal{A}}$

$$\Downarrow$$

$y_0 = 1$ we recover $\mathcal{A}$
$y_0 = 0$ we recover V.

- Local structure functions: $\left(\underbrace{(\mathcal{C}_V)^c_{ab}, \mathcal{C}^c_{0b}}_{(\mathcal{C}_{\tilde{\mathcal{A}}})^c_{ab}}, \underbrace{(\rho_V)^i_a, \rho^i_0}_{(\rho_{\tilde{\mathcal{A}}})^i_a}\right)$

$$\boxed{\begin{array}{c}\text{skew-symmetric algebroid } (\tilde{\mathcal{A}}, B_{\tilde{\mathcal{A}}}, \rho_{\tilde{\mathcal{A}}}) \\ \Updownarrow \\ \text{2-contravariant linear tensor } \Pi_{\mathcal{A}^+} \text{ in } \mathcal{A}^+\end{array}}$$

$$\begin{aligned}\Pi_{\mathcal{A}^+} \quad = \quad & \rho_0^i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_0} + \rho_b^i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_a} - B_{0b}^c p_c \frac{\partial}{\partial p_0} \wedge \frac{\partial}{\partial p_b} \\ & - \frac{1}{2} (B_V)_{ab}^c p_c \frac{\partial}{\partial p_a} \wedge \frac{\partial}{\partial p_b}\end{aligned}$$

where (x^i, p_a, p_0) are the induced coordinates on $\mathcal{A}^+$.

In our case: $\widetilde{\mathcal{A}} = V \times \mathbb{R} \to Q$ has a (skew-symmetric) algebroid structure $(B_{\widetilde{\mathcal{A}}}, \rho_{\widetilde{\mathcal{A}}})$

- local structure functions: $\left((\mathcal{C}_V)_{ab}^c, \mathcal{C}_{0a}^c; (\rho_{\widetilde{\mathcal{A}}})_a^i, (\rho_{\widetilde{\mathcal{A}}})_0^i\right)$
- $(\rho_{\widetilde{\mathcal{A}}})_a^i = (\rho_V)_a^i$
- $((\mathcal{C}_V)_{ab}^c, (\rho_V)_a^i)$ are the local structure functions of $\tau_V : V \to Q$,
- $\mathcal{C}_{0a}^c = -c_d \beta_a^d(q) \beta_b^c(q)$
- $(\rho_{\widetilde{\mathcal{A}}})_0^i = 0.$

The affgebroid structure $(B_V, \rho_{\mathcal{A}}, D)$ is given by the usual bracket B_V defined $\tau_V : V \to Q$, the anchor map $\rho_{\mathcal{A}}$ which, in this case, is equal to ρ_V and $D : \Gamma(\mathcal{A}) \times \Gamma(V) \to \Gamma(V)$ such that

$$D_{(\sigma_a + 1_{\mathcal{A}})} \sigma_b = [\![\sigma_a + \sigma_0, \sigma_b]\!]_{\mathcal{A}} = (\mathcal{C}_V)_{ab}^c \sigma_c + \mathcal{C}_{0b}^c \sigma_c .$$

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- $(\rho_{\widetilde{\mathcal{A}}})^i_0 = 0.$

The affgebroid structure $(B_V, \rho_{\mathcal{A}}, D)$ is given by the usual bracket B_V defined $\tau_V : V \to Q$, the anchor map $\rho_{\mathcal{A}}$ which, in this case, is equal to ρ_V and $D : \Gamma(\mathcal{A}) \times \Gamma(V) \to \Gamma(V)$ such that

$$D_{(\sigma_a + 1_{\mathcal{A}})} \sigma_b = [\![\sigma_a + \sigma_0, \sigma_b]\!]_{\mathcal{A}} = (\mathcal{C}_V)^c_{ab} \sigma_c + \mathcal{C}^c_{0b} \sigma_c .$$

- The ***Hamiltonian section*** h in $\Gamma(\mu_{\mathring{A}^+})$ is

$$h(x^i, p_a) = (x^i, p_a, -H(x^i, p_a)).$$

- ***The Hamiltonian vector field***

$$\mathcal{H}_h^{\Pi_{\mathcal{A}^+}}(f) \circ \mu_{\mathcal{A}^+} = \{F_h, f \circ \mu_{\mathcal{A}^+}\}_{\Pi_{\mathcal{A}^+}}, \quad f \in C^\infty(M)$$

where $F_h = -H(x^i, p_a) - p_0$

- ***Hamilton equations***

$$\begin{aligned} \dot{x}^i &= -(\rho_V)^i_a \frac{\partial H}{\partial p_a} = -(\rho_V)^i_a p_a, \\ \dot{p}_b &= -(\rho_V)^i_b \frac{\partial H}{\partial x^i} + (\mathcal{C}_V)^c_{ab} p_c \frac{\partial H}{\partial p_a} + \mathcal{C}^c_{0b} p_c, \end{aligned}$$

- ***the dissipation term:***
$\dot{H} = \{F_h, H \circ \mu_{\mathcal{A}^+}\}_{\mathcal{A}^+} = C^c_{0b} p_c \frac{\partial H}{\partial p_b} = -[c_d \beta^d_a(q) \beta^c_b(q)] p_c p_b.$