Unified formalism for implicit optimal control problems[1]

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International Young Researchers Workshop on
Geometry, Mechanics and Control
Continuous and Discrete Geometric Mechanics
Barcelona, 16 December 2008.

[1] This is a joint work between
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Geometric framework

Extended momentum phase space — *Restricted momentum phase space*

Canonical forms $\Theta,\ \Omega$

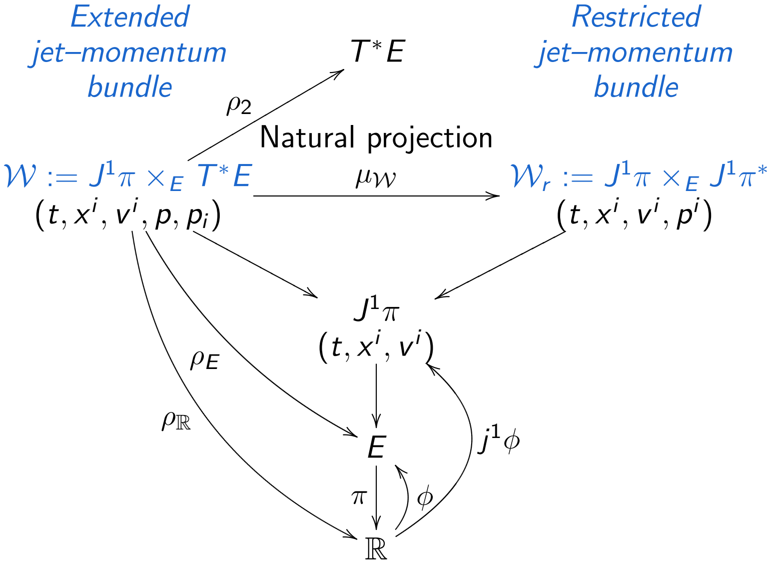
$$T^*E \;(t, x^i, p, p_i) \longrightarrow J^1\pi^* := T^*E/\pi^*T^*\mathbb{R} \;(t, x^i, p^i)$$

$$T^*E \downarrow E \;(t, x^i) \xrightarrow{\ \pi\ } \mathbb{R} \;(t)$$

Configuration fiber bundle

$\dim E = n + 1$

Skinner–Rusk unified formalism



Canonical structures on $\mathcal{W}$

- *Coupling 1–form* $\hat{\mathcal{C}}$: for every $w = (j^1\phi(t), \alpha) \in \mathcal{W}$ s.t.,

$$\hat{\mathcal{C}}(V) := \alpha(T_w(\phi \circ \pi \circ \rho_E)V), \quad V \in T_w\mathcal{W}.$$

- *Canonical forms on* $\mathcal{W}$ are
  - $\Theta_{\mathcal{W}} = \rho_2^*\Theta \in \Omega^1(\mathcal{W})$,
  - $\Omega_{\mathcal{W}} := -\mathrm{d}\Theta_{\mathcal{W}} = \rho_2^*\Omega \in \Omega^2(\mathcal{W})$.

$(\mathcal{W}, \Omega_{\mathcal{W}})$ is a presymplectic manifold.

Local expressions: $\hat{\mathcal{C}} = (\mathbf{p} + \mathbf{p_i v^i})\mathrm{d}\mathbf{t}$,

$$\Theta_{\mathcal{W}} = p_i \mathrm{d}x^i + p\mathrm{d}t, \quad \Omega_{\mathcal{W}} = -\mathrm{d}p_i \wedge \mathrm{d}x^i - \mathrm{d}p \wedge \mathrm{d}t.$$

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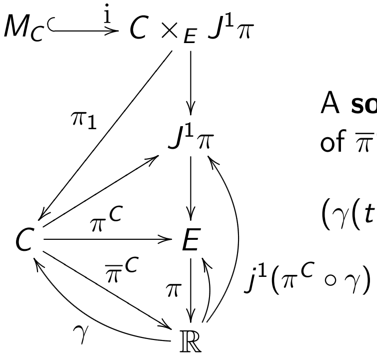
Geometric framework for implicit control systems (ICS)

A fiber bundle structure $\pi^C \colon C \longrightarrow E$.

Let C be the bundle of controls, local coordinates (t, x^i, u^l).

Let M_C be a submanifold of $C \times_E J^1\pi$.



A **solution of the ICS** is a section γ of $\overline{\pi}^C$ such that

$$(\gamma(t), j^1((\pi^C \circ \gamma)(t))) \in M_C \quad \forall\, t \in I.$$

Implicit optimal control problem

Let $\mathcal{F}\colon C \to \mathbb{R}$ be the cost function.

Given $x_0, x_f \in E$, M_C.

Find a $\mathcal{C}^2$–piecewise smooth curve $\gamma\colon I \to C$ and the final time $T \in \mathbb{R}^+$ such that

(1) $\pi^C(\gamma(0)) = x_0$, $\pi^C(\gamma(T)) = x_f$ (endpoint conditions);

(2) $(\gamma(t), j^1((\pi^C \circ \gamma)(t))) \in M_C \quad \forall\, t \in I$, solution of **ICS**;

(3) $\mathcal{S}[\gamma] = \int_0^T \mathcal{F}(\gamma(t))\,\mathrm{d}t$ is minimum among all γ satisfying (1) and (2).

Implicit optimal control problem

An **implicit optimal control problem** is given by $(\mathbf{F}, M_C)$,

- $\mathbf{F} \in \Omega^1(M_C)$, $\mathbf{F} = \mathcal{F}\mathrm{d}t$.

  $\mathcal{F} \in \mathcal{C}^\infty(C)$ is the cost function.

  ABUSE OF NOTATION: $\mathcal{F}$ should be

$$(\mathrm{i} \circ \pi_1)^* \mathcal{F} \in \mathcal{C}^\infty(M_C).$$

- M_C is a submanifold of $C \times_E J^1\pi$.

Thus, the local state equations defining M_C are

$$\Psi^k(t, x, v, u) = 0 \ , \ 1 \leq k \leq s \ , \ \text{with } \mathrm{d}\Psi^1 \wedge \ldots \wedge \mathrm{d}\Psi^s \neq 0.$$

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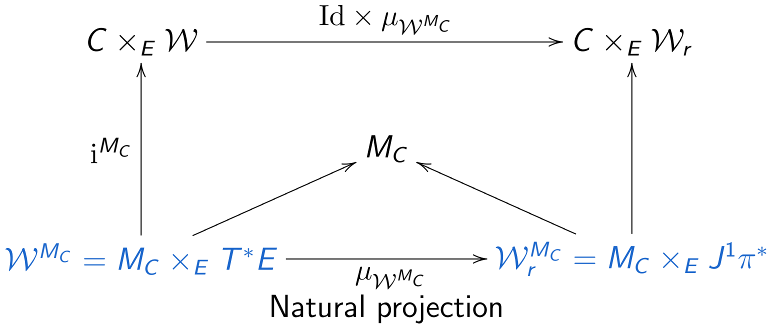
Unified formalism for implicit OCP



Extended control–jet–momentum manifold

Restricted control–jet–momentum manifold

Canonical structures on $\mathcal{W}^{M_C}$

As

$$\mathcal{W}^{M_C} \xhookrightarrow{\mathrm{i}^{M_C}} C \times_E \mathcal{W} \xrightarrow{\sigma_{\mathcal{W}}} \mathcal{W}\,,$$

the corresponding canonical structures on $\mathcal{W}^{M_C}$ are

$$\begin{aligned} \mathcal{C}_{\mathcal{W}^{M_C}} &= (\sigma_{\mathcal{W}} \circ \mathrm{i}^{M_C})^* \hat{\mathcal{C}}, & \hat{\mathcal{C}} &= (p + p_i v^i)\mathrm{d}t. \\ \Theta_{\mathcal{W}^{M_C}} &= (\sigma_{\mathcal{W}} \circ \mathrm{i}^{M_C})^* \Theta_{\mathcal{W}}, & &\Theta_{\mathcal{W}}. \\ \Omega_{\mathcal{W}^{M_C}} &= (\sigma_{\mathcal{W}} \circ \mathrm{i}^{M_C})^* \Omega_{\mathcal{W}}, & &\Omega_{\mathcal{W}}. \end{aligned}$$

Unified Pontryagin's Hamiltonian

Pontryagin's Hamiltonian is the **unique** $H_{\mathcal{W}^{M_C}} : \mathcal{W}^{M_C} \to \mathbb{R}$ s.t.

$$\mathcal{C}_{\mathcal{W}^{M_C}} - (\rho_1^{M_C})^* \mathbf{F} = H_{\mathcal{W}^{M_C}} \mathrm{d}t \ ,$$

with the natural projection $\rho_1^{M_C} : \mathcal{W}^{M_C} \longrightarrow M_C$.

Example: If $\Psi^i = v^i - f^i(t, x, u) = 0$, then locally

$$H_{\mathcal{W}^{M_C}} = p + p_i f^i - \mathcal{F}.$$

We only study **normal extremals**.

From classical Pontryagin's Maximum Principle (PMP)

As the final time is an unknown, $H_{\mathcal{W}^{M_C}} = 0$ along a solution,

$$\mathcal{W}_0^{M_C} = \left\{ w \in \mathcal{W}^{M_C} \mid H_{\mathcal{W}^{M_C}}(w) = 0 \right\} .$$

Then, there exists the natural embedding

$$\jmath_0^{M_C} : \mathcal{W}_0^{M_C} \longrightarrow \mathcal{W}^{M_C}$$

Example: If $\Psi^i = v^i - f^i(t, x, u) = 0$, then locally

$$\jmath_0^{M_C}(t, x^i, u^I, p_i) = (t, x, u, \mathcal{F}(t, x, u) - p_i f^i(t, x, u), p_j).$$

Proposition

$\mathcal{W}_0^{M_C}$ is a codimension 1, $\mu_{\mathcal{W}^{M_C}}$–transverse submanifold of $\mathcal{W}^{M_C}$, diffeomorphic to $\mathcal{W}_r^{M_C}$.

Canonical form on $\mathcal{W}_0^{M_C}$: $\Omega_{\mathcal{W}_0^{M_C}} = \left(\jmath_0^{M_C}\right)^* \Omega_{\mathcal{W}^{M_C}}$.

Let $(T_w\mathcal{W}_0^{M_C})^0$ be the annihilator of $T_w\mathcal{W}_0^{M_C}$.

Proposition

For a given $w \in \mathcal{W}_0^{M_C}$, the following conditions are equivalent:

1. *there exists a vector $Z_w \in T_w\mathcal{W}_0^{M_C}$ verifying that*

$$\Omega_{\mathcal{W}_0^{M_C}}(Z_w, Y_w) = 0 \ , \quad \text{for every } Y_w \in T_w\mathcal{W}_0^{M_C} \ ;$$

2. *there exists a vector $Z_w \in T_w(C \times_E \mathcal{W})$ satisfying*

   (I) $Z_w \in T_w\mathcal{W}_0^{M_C}$ *and*

   (II) $\mathrm{i}_{Z_w}(\sigma_{\mathcal{W}}^*\Omega_{\mathcal{W}})_w \in (T_w\mathcal{W}_0^{M_C})^0$.

Item 2 in the previous Proposition can be rewritten as follows:

There exists $Z \in \mathfrak{X}(C \times_E \mathcal{W})$ such that

(I) Z is tangent to $\mathcal{W}_0^{M_C}$ and

(II) the 1–form

$$\mathrm{i}_Z \left(\sigma_{\mathcal{W}}^* \Omega_{\mathcal{W}}\right)$$

is null on the vector fields tangent to $\mathcal{W}_0^{M_C}$.

In other words, there exist $\mu_\alpha, \mu \in \mathcal{C}^\infty(C \times_E \mathcal{W})$ such that

$$(\mathrm{i}_Z \sigma^*_{\mathcal{W}} \Omega_{\mathcal{W}})\,|_{\mathcal{W}_0^{M_C}} = (\mu_k \mathrm{d}\Psi^k + \mu \mathrm{d}(H_{\mathcal{W}^{M_C}}))|_{\mathcal{W}_0^{M_C}} \ .$$

The functions μ_k's and μ are called **Lagrange multipliers.**

In local coordinates $(t, x^i, u^l, v^i, p, p^i)$ in $C \times_E \mathcal{W}$, find

$$Z = \frac{\partial}{\partial t} + A^i \frac{\partial}{\partial x^i} + B^l \frac{\partial}{\partial u^l} + C^i \frac{\partial}{\partial v^i} + D_i \frac{\partial}{\partial p_i} + E \frac{\partial}{\partial p} \ ,$$

where A^i, B_l, C^i, D_i, E are unknown functions in $\mathcal{W}_0^{M_C}$ satisfying the following equations of motion.

Equations of motion

For $i = 1, \ldots, n$;

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu_k(t)\frac{\partial \Psi^k}{\partial v^i}(t, x(t), \dot{x}(t), u(t))\right) + \frac{\partial \mathcal{F}}{\partial x^i}(t, x(t), u(t))&\\ -\mu_k(t)\frac{\partial \Psi^k}{\partial x^i}(t, x(t), \dot{x}(t), u(t)) &= 0,\end{aligned}$$

The constrained Euler–Lagrange equations for $\mathcal{F} - \mu_k \Psi^k$.

Equations of motion

For $i = 1, \ldots, n$;

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu_k(t)\frac{\partial \Psi^k}{\partial v^i}(t, x(t), \dot{x}(t), u(t))\right) + \frac{\partial \mathcal{F}}{\partial x^i}(t, x(t), u(t)) \\ -\mu_k(t)\frac{\partial \Psi^k}{\partial x^i}(t, x(t), \dot{x}(t), u(t)) &= 0, \\ \frac{\partial \mathcal{F}}{\partial u^l}(t, x(t), u(t)) - \mu_k(t)\frac{\partial \Psi^k}{\partial u^l}(t, x(t), \dot{x}(t), u(t)) &= 0,\end{aligned}$$

Necessary condition for extremizing the Hamiltonian over the controls in the interior set.

Equations of motion

For $i = 1, \ldots, n$; $k = 1, \ldots, s$;

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu_k(t)\frac{\partial \Psi^k}{\partial v^i}(t, x(t), \dot{x}(t), u(t))\right) + \frac{\partial \mathcal{F}}{\partial x^i}(t, x(t), u(t)) & \\ -\mu_k(t)\frac{\partial \Psi^k}{\partial x^i}(t, x(t), \dot{x}(t), u(t)) &= 0, \\ \frac{\partial \mathcal{F}}{\partial u^l}(t, x(t), u(t)) - \mu_k(t)\frac{\partial \Psi^k}{\partial u^l}(t, x(t), \dot{x}(t), u(t)) &= 0, \\ \Psi^k(t, x(t), \dot{x}(t), u(t)) &= 0. \end{aligned}$$

State equations for the implicit optimal control problem.

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An OCP for a descriptor system, [Müller, 1998]

For $a_i, b_i \geq 0$ and $r > 0$, consider the problem of minimizing the functional

$$\mathcal{S} = \frac{1}{2}\int_0^{+\infty} \left[a_i(x^i)^2 + ru^2\right] \, \text{ṭ},$$

$1 \leq i \leq 3$, with control equations

$$\dot{x}^2 = x^1 + b_1 u \quad , \quad \dot{x}^3 = x^2 + b_2 u \quad , \quad 0 = x^3 + b_3 u.$$

An OCP for a descriptor system, [Müller, 1998]

$E = \mathbb{R} \times \mathbb{R}^3$ with coordinates (t, x^i).

$C = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$ with coordinates (t, x^i, u).

The submanifold $M_C \subset C \times_E J^1\pi$ is given by

$$M_C = \{(t, x^1, x^2, x^3, v^1, v^2, v^3, u) \in C \times_E J^1\pi \mid v^2 = x^1 + b_1 u, \quad v^3 = x^2 + b_2 u, 0 = x^3 + b_3 u\}.$$

$$\mathcal{W}^{M_C} = M_C \times_E T^*E, \quad \mathcal{W}^C = C \times_E J^1\pi \times_E T^*E.$$

An OCP for a descriptor system, [Müller, 1998]

The vanishing of Pontryagin's Hamiltonian defines:

$$\mathcal{W}_0^{M_C} = \{(t, x^1, x^2, x^3, v^1, v^2, v^3, u, p, p_1, p_2, p_3) \in \mathcal{W}^C \mid v^2 = x^1 + b_1 u,$$

$$v^3 = x^2 + b_2 u,\ x^3 + b_3 u = 0,\ \overbrace{p + p_1 v^1 + p_2 v^2 + p_3 v^3 - \mathcal{F} = 0}^{H_{\mathcal{W}^C}}\}.$$

Find $Z \in \mathfrak{X}\left(\mathcal{W}^C\right)$ such that

$$\begin{aligned} \mathrm{i}_Z \Omega_{\mathcal{W}^C} &= \mu_1 \mathrm{d}(x^1 + b_1 u - v^2) + \mu_2 \mathrm{d}(x^2 + b_2 u - v^3) \\ &+ \mu_3 \mathrm{d}(x^3 + b_3 u) + \mu \mathrm{d} H_{\mathcal{W}^C}\ , \end{aligned}$$

$$\begin{array}{ll} Z(x^1 + b_1 u - v^2) = 0, & Z(x^2 + b_2 u - v^3) = 0, \\ Z(x^3 + b_3 u) = 0, & Z(H_{\mathcal{W}^C}) = 0, \end{array}$$

where $\Omega_{\mathcal{W}^C} \in \Omega^2\left(\mathcal{W}^C\right)$ is the 2–form with local expression

$$\Omega_{\mathcal{W}^C} = \mathrm{d}x^1 \wedge \mathrm{d}p_1 + \mathrm{d}x^2 \wedge \mathrm{d}p_2 + \mathrm{d}x^3 \wedge \mathrm{d}p_3 + \mathrm{d}t \wedge \mathrm{d}p\ .$$

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THANK YOU!!!!!