Hamiltonization in nonholonomic Mechanics

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International Young Researchers Workshop
on Geometry, Mechanics and Control.

UPC Barcelona,
December 16-18, 2008.

Joint with Simon Hochgerner.

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- Program for today: Study the "almost" symplectic counterpart.

Outline

1. Almost Hamiltonian Systems
    - Hamiltonization, Chaplygin's reducibility theorem
2. Toy Example: The nonholonomic particle
3. Almost Symplectic Formulation of Chaplygin Nonholonomic Systems
    - Nonholonomic Mechanical Systems
    - Compression of G-Chaplygin Systems
4. The rolling ball
    - Interlude: Moving frames for $SO(3)$
    - Standard almost symplectic formulation of the Rolling ball
    - Hamiltonization of the rolling ball
5. Conclusions

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Almost Hamiltonian System

- Triple $(T^*S, \Omega_{\mathrm{as}}, \mathcal{H}_c)$.
- Ω_{as} is an *almost symplectic two-form*: non-degenerate but not necessarily closed.
- Almost Hamiltonian system:

$$\mathbf{i}_{X_{\mathcal{H}_c}} \Omega_{\mathrm{as}} = d\mathcal{H}_c.$$

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- No Arnold-Liouville theorem for integrable systems.

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- Our situation:

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Ω_S is the canonical symplectic form on T^*S.
η is a semi-basic two-form.

$$\Omega_{\mathrm{as}} = ds^i \wedge dm_i - \sum_{i<j} \eta_{ij}^k(s)\, m_k\, ds^i \wedge ds^j$$

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- Does there exist $f : S \to \mathbb{R}$, $f > 0$, such that

$$d(f\,\Omega_{\text{as}}) = 0\,?$$

We say that Ω_{as} is conformally symplectic and that f is a conformal factor.

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- *Hamiltonization*: The system is Hamiltonian in the new time and with respect to the new momenta.

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Theorem (Chaplygin's Reducibility Theorem)

Suppose that:

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- Fedorov, Jovanović (2004) important example of multi-dimensional nonholonomic system that can be written in Hamiltonian form.

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- Particle in $\mathbb{R}^3$ with nonholonomic constraint $\dot{z} = y\dot{x}$. (Constraint one-form $\epsilon = dz - ydx$.)

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$$\mathcal{L}_c(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\left((1 + y^2)\dot{x}^2 + \dot{y}^2\right)$$

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$$\eta = p_z\, d\epsilon \qquad (\text{ Momentum } \times \text{ Curvature })$$

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- Look for a conformal factor $f = f(y)$. Enforcing $d(f\,\Omega_{\mathrm{nh}}) = 0$ yields

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$$f(y) = \frac{1}{\sqrt{1+y^2}}$$
is a conformal factor and
$$\mu = \frac{1}{\sqrt{1+y^2}}\, dx \wedge dy \wedge dp_x \wedge dp_y$$
is a preserved measure.

The nonholonomic particle - continued

- Define new time and new momenta:

$$d\tau = \frac{dt}{\sqrt{1+y^2}} \qquad \tilde{m}_x = \frac{m_x}{\sqrt{1+y^2}} \quad \tilde{m}_y = \frac{m_y}{\sqrt{1+y^2}}$$

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- Compressed equations of motion:

$$\frac{dx}{d\tau} = \frac{\partial \mathcal{H}_c}{\partial \tilde{m}_x} \qquad \frac{dy}{d\tau} = \frac{\partial \mathcal{H}_c}{\partial \tilde{m}_y}$$

$$\frac{d\tilde{m}_x}{d\tau} = -\frac{\partial \mathcal{H}_c}{\partial \tilde{x}} \qquad \frac{d\tilde{m}_y}{d\tau} = -\frac{\partial \mathcal{H}_c}{\partial y}$$

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$$\epsilon^a_i(q)\dot{q}^i = 0, \qquad a = 1, \dots, k.$$

$\mathcal{D}_q \subset T_qQ$ is the annihilator of the one-forms on Q:

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- Conservation of Energy. Lagrange Multipliers λ_i.

Hamiltonian Formalism

- Generalized momenta $p_s = \frac{\partial \mathcal{L}}{\partial \dot{q}^s}$. Legendre transform: Leg : $TQ \to T^*Q$.
- Hamiltonian $\mathcal{H} : T^*Q \to \mathbb{R}$.
- Constraint submanifold $\mathcal{M} = \text{Leg}(\mathcal{D}) \subset T^*Q$,

$$\mathcal{M} = \left\{ (p,q) \; : \; \sum_{i=1}^{n} \epsilon_i^a(q) \, \frac{\partial \mathcal{H}}{\partial p_i} = 0, \quad a = 1, \dots, k \right\}.$$

- Equations of motion

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} + \sum_{a=1}^{k} \lambda_a \epsilon_i^a(q).$$

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- $\mathcal{C}$ is a non-integrable distribution on $\mathcal{M}$.

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The pointwise restriction of $\iota^*\Omega_Q$ to $\mathcal{C}$, denoted $\Omega_{\mathcal{C}}$, is non-degenerate. (i.e. $\mathcal{C}_m$ is a symplectic subspace of $T_m(T^*Q)$).

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- Restriction of the intrinsic equations to $\mathcal{C}$ gives

$$\mathbf{i}_{X_{\mathrm{nh}}^{\mathcal{M}}} \Omega_{\mathcal{C}} = (d\mathcal{H})_{\mathcal{C}}$$

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- Starting point to write the equations as an *almost Hamiltonian system*

$$\mathbf{i}_{X_{\mathrm{nh}}} \Omega_{as} = d\mathcal{H}_c$$

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1. Almost Hamiltonian Systems
   - Hamiltonization, Chaplygin's reducibility theorem

2. Toy Example: The nonholonomic particle

3. Almost Symplectic Formulation of Chaplygin Nonholonomic Systems
   - Nonholonomic Mechanical Systems
   - Compression of G-Chaplygin Systems

4. The rolling ball
   - Interlude: Moving frames for $SO(3)$
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- Examples: rolling systems. $G = \mathbb{R}^2$.
- The space $S := Q/G$ is called *shape space*.
- First studied by Chaplygin around 1895, later by Koiller (non-abelian case) around 1992. Main result, group variables can be "eliminated". Second order system on S. Almost Hamiltonian system on T^*S.

Ingredients for Compression

- The splitting

$$T_qQ = \mathcal{D}_q \oplus T_q\mathrm{Orb}_G(q)$$

defines a principal connection $\mathcal{A} : TQ \to \mathfrak{g}$ on the principal bundle $\pi : Q \to S$.

$$\begin{aligned} \mathcal{D}_q = \mathrm{hor}_q &= \{v_q \in T_qQ \, : \, \mathcal{A}(v_q) = 0\} \\ T_q\mathrm{Orb}_G(q) = \mathrm{ver}_q &= \{\xi_Q(q) \, : \, \xi \in \mathfrak{g}\} \end{aligned}$$

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- Vertical and horizontal projections:

$$\mathrm{ver}_q\, v_q = (\mathcal{A}(v_q))_Q(q), \qquad \mathrm{hor}_q\, v_q = v_q - (\mathcal{A}(v_q))_Q(q).$$

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- Horizontal Lift of $v_s \in T_sS$ at $q \in \pi^{-1}(s)$ is the unique vector $v_q \in \mathcal{D}_q$ such that
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  so $\mathcal{K}$ measures non-integrability of $\mathcal{D}$. Note: $\mathcal{K}$ is equivariant.
- Equivariant momentum map $J : TQ \to \mathfrak{g}^*$ defined by
$$\langle J(v_q), \xi \rangle = \mathbb{G}_q(v_q, \xi_Q(q)).$$

The Compressed Lagrangian

- Define the compressed Lagrangian $\mathcal{L}_c : TS \to \mathbb{R}$ by

  $$\mathcal{L}_c(v_s) = \mathcal{L}(v_q)$$

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- Compressed Legendre transform $\text{Leg}_c : TS \to T^*S$.

  Compressed Hamiltonian $\mathcal{H}_c : T^*S \to \mathbb{R}$.

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- We will be interested in the pull-back of $\langle J, \mathcal{K} \rangle$ to T^*S by Leg_c^{-1}.

The standard nonholonomic two-form and compression

- Define the standard nonholonomic two-form on T^*S by

$$\Omega_{\mathsf{nh}} = \Omega_S - \langle J, \mathcal{K} \rangle$$

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Theorem (Koiller)

Let $\rho : \mathcal{M} \to T^*S$ *be defined by the composition:*

$$\rho : \ \mathcal{M} \xrightarrow{Leg^{-1}} \mathcal{D} \xrightarrow{T\pi|_{\mathcal{D}}} TS \xrightarrow{Leg_c} T^*S$$

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& T^*Q & & &
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2. $\rho^*\Omega_{\mathrm{nh}}|_{\mathcal{C}} = \iota^*\Omega_Q|_{\mathcal{C}} := \Omega_{\mathcal{C}}$. *(However $\rho^*\Omega_{\mathrm{nh}} \neq \iota^*\Omega_Q$ on $\mathcal{M}$ in general.)*
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Almost Hamiltonian System $(T^*S, \Omega_{\mathrm{nh}}, \mathcal{H}_c)$.

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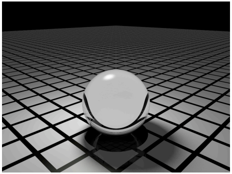
- Consider "trivial problem" homogeneous ball that rolls without slipping on the plane. (No constraint forces)



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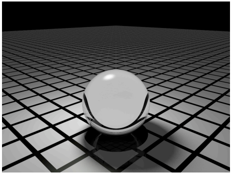
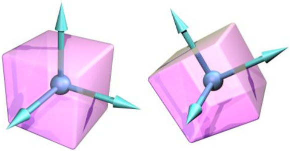


$$Q = SO(3) \times \mathbb{R}^2$$

$q = (s,(x,y)) \in Q$ where

$(x,y) \in \mathbb{R}^2$ are cartesian coordinates on the plane

$s \in SO(3)$ relates a spatial coordinate frame to a moving coordinate frame:



These frames define *space* and *body* coordinates.

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Right moving frame for $SO(3)$

- Identify the Lie algebra $\mathfrak{so}(3)$ with $\mathbb{R}^3$ using the *hat-map*,

$$\mathbf{v} = (v_1, v_2, v_3) \mapsto \hat{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

Lie algebra isomorphism ($\mathbb{R}^3$ equipped with the usual vector product).

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- Let $\{e_1, e_2, e_3\}$ be the canonical basis for the Lie algebra $\mathfrak{so}(3) = \mathbb{R}^3$.
- Moving frame by *right translation*:

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$$\{\, X_1^{\mathrm{r}}(s),\, X_2^{\mathrm{r}}(s),\, X_3^{\mathrm{r}}(s) \,\} \qquad \text{basis for } T_s SO(3).$$

- Dual co-frame

$$\{\, \rho_1(s),\, \rho_2(s),\, \rho_3(s) \,\} \qquad \text{basis for } T_s^* SO(3).$$

- Components of a tangent vector $v_s \in T_sSO(3)$:

$$v_s = \omega_1 X_1^{\mathrm{r}}(s) + \omega_2 X_2^{\mathrm{r}}(s) + \omega_3 X_3^{\mathrm{r}}(s)$$

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  $\Pi = (\Pi_1, \Pi_2, \Pi_3)$ is angular momentum in space coordinates. (Have in mind rigid body kinetic energy $\mathcal{L} = \frac{1}{2}\langle \alpha_s, v_s \rangle = \frac{1}{2}(\Pi \cdot \omega)$).

- Note: $\{\tau^* \rho_i, d\Pi_i\}$ is a moving co-frame for $T^*SO(3)$. Canonical one-form:

$$\Theta = \Pi_i \, \rho_i$$

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- Using Cartan's structure equations $d\rho_1 = \rho_2 \wedge \rho_3, \dots$ canonical 2-form:

$$\Omega = \rho_i \wedge d\Pi_i - \Pi_1 \, \rho_2 \wedge \rho_3 - \Pi_2 \, \rho_3 \wedge \rho_1 - \Pi_3 \, \rho_1 \wedge \rho_2$$

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Rolling ball continued

- Write a tangent vector $v_q \in T_qQ$ as

$$v_q = \omega_i X_i^{\mathrm{r}}(s) + v_x\partial_x + v_y\partial_y$$

$q = (s, (x, y)) \in Q = SO(3) \times \mathbb{R}^2.$

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- Kinetic energy Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(I\omega\cdot\omega\right) + \frac{m}{2}\left(v_x^2 + v_y^2\right)$$

Rolling ball continued

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- Distribution:

$$\mathcal{D} = \text{span}\left\{\, X_1^{\mathrm{r}} - r\partial_y,\, X_2^{\mathrm{r}} + r\partial_x,\, X_3^{\mathrm{r}} \,\right\}$$

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- Curvature $\mathcal{K}$:

$$\mathcal{K} = r\,(\rho_1 \wedge \rho_3, \rho_2 \wedge \rho_3)$$

Rolling ball continued

- Horizontal lift of $v_s = \omega_i X_i^{\mathrm{r}}(s)$:

$$v_q = \omega_1(X_1^{\mathrm{r}}(s) - r\partial_y) + \omega_2(X_2^{\mathrm{r}}(s) + r\partial_x) + \omega_3 X_3^{\mathrm{r}}(s)$$

- Compressed Lagrangian:

$$\mathcal{L}_c(v_s) = \frac{1}{2}\left((I + mr^2)\omega_1^2 + (I + mr^2)\omega_2^2 + I\omega_3^2\right)$$

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- Momentum map $J : TQ \to (\mathbb{R}^2)^* = \mathbb{R}^2$

$$J(v_q) = m(v_x, v_y)$$

Rolling ball continued

- $\langle J, \mathcal{K} \rangle$ two-form on TS:

$$\langle J, \mathcal{K} \rangle (v_s) = mr^2 (\omega_2 \, \rho_1 \wedge \rho_3 - \omega_1 \, \rho_2 \wedge \rho_3)$$

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- Standard nonholonomic two-form:

$$\begin{aligned} \Omega_{\text{nh}} &= \Omega_S - \langle J, \mathcal{K} \rangle \\ &= \rho_i \wedge dK_i - K_1 \, \rho_2 \wedge \rho_3 - K_2 \, \rho_3 \wedge \rho_1 - K_3 \, \rho_1 \wedge \rho_2 \\ &\qquad + \frac{mr^2}{I + mr^2} (K_2 \, \rho_3 \wedge \rho_1 + K_1 \, \rho_2 \wedge \rho_3) \end{aligned}$$

Rolling ball continued

- Compressed Hamiltonian $\mathcal{H}_c : T^*S \to \mathbb{R}$:

$$\mathcal{H}_c = \frac{1}{2}\left(\frac{K_1}{I+mr^2} + \frac{K_2}{I+mr^2} + \frac{K_3}{I}\right)$$

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- Preserved measure

$$\Omega_S^3 = \rho_1 \wedge \rho_2 \wedge \rho_3 \wedge dK_1 \wedge dK_2 \wedge dK_3$$

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- Not Hamiltonizable? Do (almost) symplectic reduction!

Almost Symplectic Point Reduction (Planas Bielsa 2004)

- Almost Hamiltonian System $(T^*S, \Omega_{\text{as}}, \mathcal{H}_c)$
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$$\Phi_h^* \Omega_{\text{as}} = \Omega_{\text{as}} \qquad \text{for all} \quad h \in H.$$

  - $\ell : T^*S \to \mathfrak{h}^*$ is a momentum map if

$$\mathbf{i}_{\xi_{T^*S}} \Omega_{\text{as}} = d\langle \ell, \xi \rangle \qquad \text{for all} \qquad \xi \in \mathfrak{h}.$$

The momentum map is said to be equivariant if

$$\mathrm{Ad}^*_{h^{-1}} \circ \ell = \ell \circ \Phi_h$$

Almost Symplectic Point Reduction

Theorem (Almost Symplectic Point Reduction)

$\Phi : H \times T^*S \to T^*S$ *free and proper canonical action.*

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- $P_\mu := \ell^{-1}(\mu)/H_\mu$ *is an almost symplectic manifold with symplectic form* Ω^μ_{as} *uniquely determined by*

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- *If* $\mathcal{H}_c$ *is H-invariant then there is a reduced Hamiltonian* $\mathcal{H}^\mu_c : P_\mu \to \mathbb{R}$ *and the corresponding almost Hamiltonian vector fields are* π_μ*-related.*

Reduction of the Rolling Ball

- Symmetry group

$$H = SO(2) = \{ \exp(te_3) \, : \, t \in \mathbb{R} \} \subset S$$

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- Last, unexpected resource:

Change Ω_{nh}!

Truncation

- Recall our freedom:

$$\mathbf{i}_{X_{\mathrm{nh}}^{\mathcal{M}}}\iota^*(\,\Omega_Q+\Omega_0)=\iota^*\left(d\mathcal{H}+\sum_{a=1}^{k}\lambda_a\tau^*\epsilon^a\right).$$

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Non-standard almost symplectic structures.
Pick the one that deviates the least from the canonical form on T^*S.

Truncation for the Rolling ball

- We had standard nonholonomic form:

$$\Omega_{\mathrm{nh}} = \Omega_S + \underbrace{mr^2 \left(\frac{K_2}{I + mr^2}\, \rho_3 \wedge \rho_1 + \frac{K_1}{I + mr^2}\, \rho_2 \wedge \rho_3 \right)}_{-\langle J, \mathcal{K} \rangle}$$

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- Replace by

$$\Omega_{\mathrm{as}} = \Omega_S - mr^2 \left(\frac{K_3}{I}\, \rho_1 \wedge \rho_2 \right)$$

Ok since $\Omega_{\mathrm{nh}} - \Omega_{\mathrm{as}}$ annihilates X_{nh}.

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- We can do almost symplectic point reduction. System reduces and reduced almost Hamiltonian system on $\ell^{-1}(K_3)$ is actually Hamiltonian!!!

$$d\Omega_{\mathrm{as}} = -\frac{mr^2}{I}\left(dK_3 \wedge \rho_1 \wedge \rho_2 \right)$$

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- Hamiltonization for $\dim S \geq 3$ is much harder. Non-standard almost symplectic structures should be considered.
- Open problems:
  - Find high dimensional Hamiltonizable systems.
  - Provide an explanation for the presence of particular magnetic term.

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