The Motion of Solid Bodies in Perfect Fluids: A Geometric Outlook

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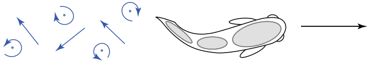
Motivation

- There are tons of situations where one would like to know the motion of a rigid body in a fluid.



- Unfortunately, the Euler equations are for all practical purposes intractable:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p,$$

  plus boundary conditions.

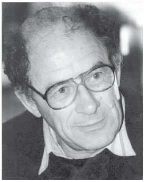
- Luckily, we can identify large-scale structures in the flow or make simplifying assumptions and use those as elementary building blocks to construct a faithful approximation.
- Potential flow, point vortices, . . .

Joint work with Eva Kanso (University of Southern California) and Jerrold E. Marsden (Caltech).

Geometric methods in fluid mechanics

- "Secret" motivation behind this talk: the use of geometry in fluid mechanics is just too elegant to ignore.
- Example: Euler equations (Euler, 1740): geodesic flow on the group of volume-preserving diffeomorphisms (Arnold, 1966).

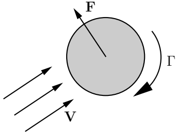
$\Longrightarrow$



Case in point: circulation

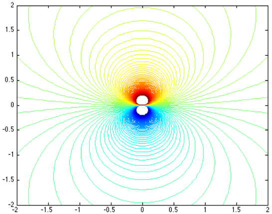
- Circulation around a rigid body: measures how many times fluid "goes round" the body.
- Kutta-Joukowski theorem:

  An airfoil in an airstream with velocity $Ve^{i\alpha}$ and circulation Γ experiences a lift force $F = \rho \Gamma V e^{-i\alpha}$.



- (Partially) Explanation of why aircraft fly.

> Underlying geometry: *The KJ force is the gyroscopic force generated by the curvature of the Neumann connection.*
> Buzzwords for now, will be explained later!



Our model

- We will study the dynamics of a circular, rigid body interacting with N point vortices with strength Γ_i;

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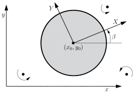
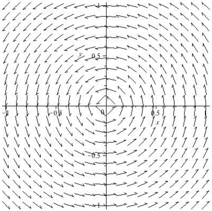
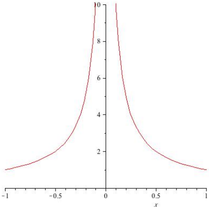


- Fluid: inviscid and incompressible.
- Point vortex flow: superposition of elementary "quanta" of rotation in the flow. Flow field of a single point vortex:





History of solid-vortex interactions

- Motion of solids in potential flow: Kirchhoff (1877); dynamics of point vortices in a bounded domain: Lin (1941).
- By contrast: motion of solids interacting with vortices: Shashikanth, Marsden, Burdick, Kelly (2002), Borisov, Mamaev, Ramodanov (2003).
- Both groups came up with an (ad-hoc) Hamiltonian description on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$:

	SMBK	**BMR**
Poisson bracket	canonical	modified
Hamiltonian	interaction-type	kinetic energy.

Different descriptions of the dynamics

BMR

Poisson brackets

$$\{v_1, v_2\} = \frac{\lambda}{\mu^2} - \sum \frac{\lambda}{\mu^2} \frac{r_i^4 - R^4}{r_i^4}, \quad \{v_1, x_i\} = \frac{1}{\mu} \frac{r_i^4 - R^2(x_i^2 - y_i^2)}{r_i^4},$$

$$\{v_1, y_i\} = -\frac{1}{\mu} \frac{2R^2 x_i y_i}{r_i^4}, \quad \{v_2, x_i\} = -\frac{1}{\mu} \frac{2R^2 x_i y_i}{r_i^4},$$

$$\{v_2, y_i\} = \frac{1}{\mu} \frac{r_i^4 + R^2(x_i^2 - y_i^2)}{r_i^4}, \quad \{x_i, y_i\} = -\frac{1}{\lambda_i}.$$

Hamiltonian

$$H = \frac{c}{2} \langle \mathbf{V}, \mathbf{V} \rangle - W_G(\mathbf{V}, \mathbf{l}_k).$$

Equations of motion

$$\dot{F} = \{F, H\}.$$

SMBK

Equations of motion

$$\frac{d\mathbf{L}}{dt} = 0, \quad \frac{d\mathbf{A}}{dt} + \mathbf{V} \times \mathbf{L} = 0, \quad \text{and} \quad \Gamma_k \frac{d\mathbf{X}_k}{dt} = -J \frac{\partial H}{\partial \mathbf{X}_k},$$

Momenta

$$\mathbf{L} = c\mathbf{V} + \sum \Gamma_k \mathbf{X}_k \times \mathbf{e}_3 + \sum \Gamma_k \mathbf{e}_3 \times \frac{\mathbf{X}_k}{\|\mathbf{X}_k\|^2}.$$

Hamiltonian

$$H(\mathbf{L}, \mathbf{X}_k) = -W(\mathbf{L}, \mathbf{X}_k) + \frac{1}{2c} \|\mathbf{X}_k\|^2 - \frac{1}{c} \Big(\sum \Gamma_k (\mathbf{L} \times \mathbf{X}_k) \cdot \mathbf{e}_3$$
$$- \frac{1}{2} \sum \Gamma_k^2 \|\mathbf{X}_k\|^2 - \sum_{j>k} \Gamma_k \Gamma_j \mathbf{X}_k \cdot \mathbf{X}_j$$
$$+ \frac{1}{2} \Big\langle \sum \Gamma_k \frac{\mathbf{X}_k}{\|\mathbf{X}_k\|^2}, \sum \Gamma_k \frac{\mathbf{X}_k}{\|\mathbf{X}_k\|^2} \Big\rangle \Big).$$

Our aim in this lecture

We will put the pioneering work of SMBK and BMR on a firm geometrical footing.

- By symplectic reduction we will obtain BMR and SMBK, and the link between both;
- The interaction between solids and point vortices is due to the curvature of a certain hydrodynamical connection.

In the process, some interesting facts arise:

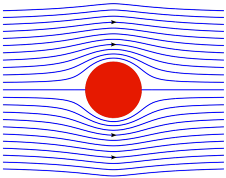
- The system is fundamentally similar to the geometric description of a charged particle in a magnetic field;
- The Kutta-Joukowski force on an airfoil is a gyroscopic force due to curvature.

Potential Flow

- If $\mathbf{u}(\mathbf{x}, t)$ is the fluid velocity, then the vorticity is defined as

$$\omega = \nabla \times \mathbf{u}.$$

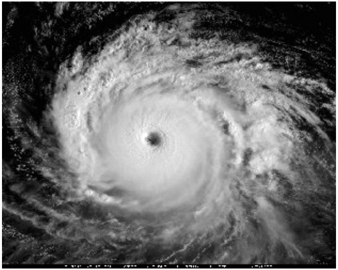
- Basic assumption: $\omega = 0$ (potential flow). Hence, $\mathbf{u} = \nabla\phi$. In other words, $\mathbf{u}$ is perpendicular to the level sets of a global function ϕ.



- This assumption is somewhat artificial. Richard Feynman referred to potential flow as "dry water". Nevertheless very good in some cases.

Vorticity in two dimensions

- Vorticity measures rotation: a paddle-wheel at $\mathbf{x}$ will rotate with angular velocity $\omega(\mathbf{x}, t)$. Hence: rotation $\implies$ no potential flow.



- Point vortices: singular distributions of vorticity: $\omega(\mathbf{x}) = \Gamma\delta(\mathbf{x} - \mathbf{x}_0)$.
- Point vortices keep their structure and are finite-dimensional flow structures characterized by their strength Γ and location $\mathbf{x}_0$.

Viscosity

Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity. Lewis Fry Richardson (1920)

- Viscosity term in the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - \nu \nabla^2 \mathbf{u}.$$

- Here, we put $\nu = 0$: no decay of vorticity. Conservation law in two dimensions:

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0.$$

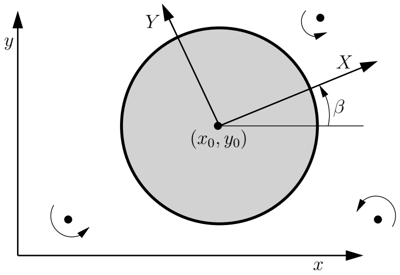
  Vorticity is advected with the flow.

- Kelvin's theorem: $\frac{d\Gamma}{dt} = 0$, where

$$\Gamma = \oint_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{l} = \int_S \omega \cdot \mathbf{n}\, \mathrm{d}S$$

Rigid bodies interacting with point vortices

- We will study the dynamics of a circular, rigid body interacting with N point vortices with strength Γ_i;
- Fluid: 2D, inviscid;
- General body shapes can be treated with conformal mapping techniques.



Outline of the method

- Consider the motion of a fluid and a rigid body as geodesic motion on

$$\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2)$$

with respect to the kinetic energy metric (Arnol'd).

- Do symplectic reduction with respect to the particle relabeling group; this imposes the condition that the vorticity is given by N point vortices.

Lamb (1896), §144

To some writers the matter has presented itself as a much simpler one. The problems are brought at one stroke under the sway of the ordinary formula of Dynamics by the imagined introduction of an infinite number of 'ignored co-ordinates,' which would specify the configuration of the various particles of the fluid. The corresponding components of momentum are assumed all to vanish, with the exception (in the case of a cyclic region) of those which are represented by the circulations through the several apertures.

Toy example: a charged particle in a magnetic field

The motion of a charged particle of mass m and charge e in a magnetic field $\mathbf{B} = \mathrm{d}\mathbf{A}$ can be determined in two equivalent ways:

- Through the minimal coupling Hamiltonian
$$H = \frac{1}{2m} \left\| \mathbf{p} - e\mathbf{A} \right\|^2 ;$$
- By using the kinetic energy $H = \left\|\mathbf{p}\right\|^2 / 2m$ and the magnetic symplectic form:
$$\Omega_{\mathbf{B}} = \Omega_{\mathrm{can}} + e\mathbf{B}.$$

Surprisingly, the relation between both has an exact analogue in the interaction of solids and vortices!

Reconciling both points of view: the Kaluza-Klein approach

- Kaluza-Klein: the trajectories of the charged particle are geodesics on $Q = M \times U(1)$.
- Let $M = \mathbb{R}^3$ with the Euclidian metric h and let $\mathbf{A}$ be a vector potential on M. Define a new metric g on Q by putting

$$g_{ab} = \begin{pmatrix} h_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\mu & 1 \end{pmatrix}.$$

- Kinetic energy Hamiltonian:

$$H = \frac{1}{2m} \left\| \mathbf{p} - p\mathbf{A} \right\|^2 + \frac{p^2}{2}.$$

Symplectic reduction recovers the minimal coupling picture

Starting point: $U(1)$ acts on Q and leaves the Hamiltonian invariant.

- Momentum map:

$$J : T^*Q \to \mathbb{R}, \quad J(\mathbf{x}, \theta, \mathbf{p}, p) = p.$$

- Reduced spaces: for $e \in \mathbb{R}$, we have $J^{-1}(e) = \{(\mathbf{x}, \theta, \mathbf{p}, e)\}$ and

$$J^{-1}(e)/U(1) = \{(\mathbf{x}, \mathbf{p})\} = T^*M,$$

  with the canonical symplectic form.
- The Hamiltonian on Q drops to T^*M:

$$H = \frac{1}{2m} \|\mathbf{p} - e\mathbf{A}\|^2 .$$

Conclusion: symplectic reduction yields the minimal coupling picture.

Shifting the symplectic form

What if we would like to work in the modified symplectic picture?

- Define a shift map $\Phi : T^*M \to T^*M$

$$\Phi : (\mathbf{x}, \bar{\mathbf{p}}) \mapsto (\mathbf{x}, \mathbf{p} = \bar{\mathbf{p}} + e\mathbf{A})$$

Note that $\Phi^* H_{\mathbf{A}} = H_{\mathrm{kin}}$.

- Φ is a symplectic map:

$$\begin{aligned} \Phi^*\Omega_{\mathrm{can}} &= \Phi^*(\mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{x}) \\ &= \mathrm{d}(\bar{\mathbf{p}} + e\mathbf{A}) \wedge \mathrm{d}\mathbf{x} \\ &= \Omega_{\mathrm{can}} + e\mathbf{B} \end{aligned}$$

Conclusion: Φ pulls back the minimal coupling picture to the modified symplectic picture.

The geometry behind all this: connections and curvature

Define the forms

$$\mathcal{A} = \mathbf{A} + d\theta \quad \text{and} \quad \mathcal{B} = \mathrm{d}\mathcal{A}.$$

- $\mathcal{A}$ is a $U(1)$-connection on Q.
- The curvature $\mathcal{B}$ is equal to the magnetic field $\mathbf{B}$.

Hamilton's equations for the modified symplectic form $\Omega_{\mathbf{B}}$:

$$i_X \Omega_{\mathbf{B}} = \mathrm{d}H \quad \Longleftrightarrow \quad i_X \Omega_{\mathrm{can}} = \mathrm{d}H - e i_X \mathbf{B}$$

The Lorentz force $e i_X \mathbf{B}$ is a gyroscopic force, due to the curvature $\mathcal{B}$.

Relation with the fluid-solid problem

	Magnetic terms	**Minimal coupling**
Symplectic form	curvature	canonical
Hamiltonian	canonical	connection form.

- For the solid-fluid system, G will be the group $\mathrm{Diff}_{\mathrm{vol}}$ of volume-preserving diffeomorphisms. The magnetic picture will correspond to Borisov *et al.* and the minimal coupling picture to Shashikanth *et al.*
- The analogue of the Lorentz force will be the Kutta-Joukowski force on a rigid body with circulation.

Dictionary

Charged particle	**Fluid-solid problem**
$U(1)$	$\mathrm{Diff}_{\mathrm{vol}}$
charge e	vorticity ω
vector potential **A**	connection form $\mathcal{A}$
magnetic field **B**	magnetic two-form β_μ
Lorentz force	Kutta-Joukowski force
Minimal coupling	BMR equations
Modified symplectic picture	SMBK equations

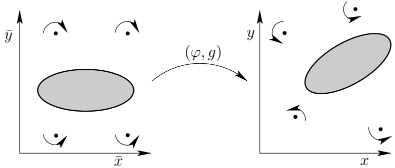
The fluid-solid configuration space

We label each fluid particle by its initial location, and track its motion as a function of time. If the initial label space is denoted by $\mathcal{F}_0$, then the configuration space is hence

$$\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2),$$

with the following conditions:

- for $\phi \in \mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2)$, we have $\phi^* d^2\mathbf{x} = d^2\mathbf{x}_{\mathcal{F}_0}$ (volume preservation);
- The image of φ together with the rigid body fills up $\mathbb{R}^2$ (no cavitation/interpenetration, slip boundary condition).



The tangent space to $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2)$

Consider an element $(\varphi, g) \in \mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2)$. How can we describe the tangent space $T_{(\varphi,g)}(\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2))$?

- Take a curve $t \mapsto (\varphi_t, g_t)$ such that $\varphi_0 = \varphi$ and $g_0 = g$.
- The derivative $\dot{\varphi}_0$ is a map from $\mathcal{F}_0$ to $T\mathbb{R}^2$, defined as

$$\dot{\varphi}_0(x) = \left.\frac{\partial \varphi_t(x)}{\partial t}\right|_{t=0} \in T_{\varphi(x)}\mathbb{R}^2.$$

  In other words, $\dot{\varphi}_0$ is a vector field along φ.

The kinetic energy

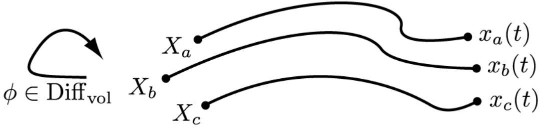
Let $(\varphi, g; \dot\varphi, \dot g)$ be an element of $T(\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2))$ and define $\mathbf{u} = \dot\varphi \circ \varphi^{-1}$ and $(\Omega, \mathbf{V}) = g^{-1}\dot g$.

$$T = \underbrace{\frac{\rho}{2}\int_{\mathcal{F}} \left\|\mathbf{u}\right\|^2 \, d^2\mathbf{x}}_{\text{fluid}} + \underbrace{\frac{\mathbb{I}}{2}\Omega^2 + \frac{m}{2}\mathbf{V}^2}_{\text{body}}.$$

- The system describes a geodesic on $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2)$ with respect to the kinetic-energy metric.
- This is valid for arbitrary distributions of vorticity.
- To bring in point vortices: symplectic reduction.

Particle relabeling symmetry

- The physics is *indifferent to the way in which we label the individual fluid particles*.



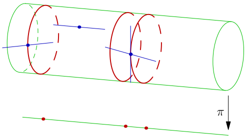
- The group $\mathrm{Diff}_{\mathrm{vol}}$ of volume-preserving diffeomorphisms acts on the right on $\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2)$:

$$(\varphi, g) \cdot \phi = (\varphi \circ \phi, g),$$

  leaving the kinetic energy invariant.
- The projection $\pi : \mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2) \to SE(2)$ is a principal fiber bundle with structure group $\mathrm{Diff}_{\mathrm{vol}}$.

Interlude: bundles and connections

- Let G be a group acting on a manifold Q so that Q/G is again a manifold. Then $\pi : Q \to Q/G$ is termed a principal fiber bundle with structure group G.



- A connection on Q is one of the following:
  1. A G-equivariant one-form $\mathcal{A} : TQ \to \mathfrak{g}^*$ such that
  $$\mathcal{A}(v_q \cdot g) = \mathrm{Ad}^*_{g^{-1}} \mathcal{A}(v_q) \quad \text{and} \quad \mathcal{A}(\xi_Q) = \xi$$
  for all $\xi \in \mathfrak{g}$.
  2. A G-invariant distribution H such that $TQ = H \oplus V\pi$.

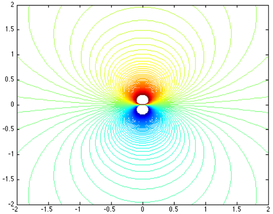
What happens if we move the rigid body?

- For each infinitesimal motion $(g, \dot{g}) \in TSE(2)$ of the rigid body, consider the vector field $\nabla\Phi$, where Φ solves

$$\nabla^2\Phi = 0 \quad \text{and} \quad \frac{\partial\Phi}{\partial n} = (\mathbf{V} + \Omega \times \mathbf{X}) \cdot \mathbf{n} \quad \text{on } \partial\mathcal{F}$$

  and $(\Omega, \mathbf{V}) = g^{-1}\dot{g}$.
- $\nabla\Phi$ gives the response of the fluid to the rigid body motion $(g, \dot{g})$.
- Example: unit speed motion in x-direction.

$$\Phi = R\frac{X}{X^2 + Y^2}$$



Making this rigourous: the Neumann connection

- For all (φ, g), define a horizontal lift mapping

$$\mathbf{h}_{(\varphi,g)} : (g, \dot{g}) \mapsto \nabla\Phi \circ \varphi,$$

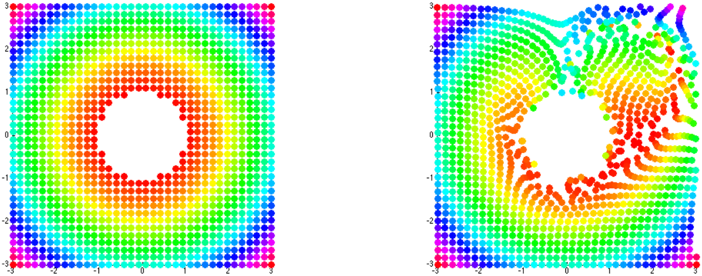
- Define $H(\varphi, g) = \mathrm{Im}\,\mathbf{h}_{(\varphi,g)}$; this is a principal fiber bundle connection called the Neumann connection.
- The connection form $\mathcal{A}$ is given by the Helmholtz-Hodge decomposition:

$$\mathcal{A}(\varphi, g, \dot{\varphi}, \dot{g}) = \varphi^* \mathbf{u}_V \in \mathfrak{X}^*_{\mathrm{vol}},$$

where $\mathbf{u} = \nabla\Phi + \mathbf{u}_V$.

The curvature of the Neumann connection

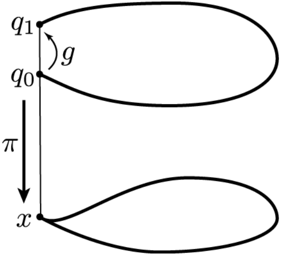
- Question: move the rigid body around in a closed loop. Will the fluid particles also return to their original location?



- Failure of fluid particles to return to their original locations: geometric phase due to curvature of the Neumann connection.
- Classically known as Darwin drift (1953).

Interlude: geometric phases

- Use the connection to lift closed path $\sigma(t)$ in Q/G to a path $\sigma^H(t)$ in Q.
- Failure of $\sigma^H(t)$ to close: geometric phase.
- Ambrose-Singer: “infinitesimal” geometric phases are generated by the curvature $\mathcal{B}$.
- No curvature $\implies$ no geometric phase.



Appearance in general relativity (parallel transport), quantum mechanics (Berry's phase, holonomic quantum computing), integrable systems (Hannay's angles), classical mechanics (falling cat, Foucault pendulum), etc.

The Lie algebra of $\mathrm{Diff}_{\mathrm{vol}}$...

- The space $\mathrm{Diff}_{\mathrm{vol}}$ is (formally) a Lie group: multiplication is given by composition, and the unit is the identity map:

$$\phi \cdot \psi = \phi \circ \psi \quad \text{and} \quad e = \mathrm{id}.$$

- Lie algebra: $T_e\mathrm{Diff}_{\mathrm{vol}} = \mathfrak{X}_{\mathrm{vol}}$, the algebra of divergence-free vector fields on $\mathcal{F}_0$ which are tangent to $\partial\mathcal{F}_0$, with the Jacobi bracket:

$$[X, Y]_{\mathfrak{X}} = -[X, Y].$$

The minus sign is due to the *right* action of $\mathrm{Diff}_{\mathrm{vol}}$ on itself.

...and its dual $\mathfrak{X}^*_{\mathrm{vol}}$

- An naive candidate for $\mathfrak{X}^*_{\mathrm{vol}}$ is $\Omega^1(\mathcal{F}_0)$, with pairing

$$\langle \alpha, X \rangle = \int_{\mathcal{F}_0} \alpha(X)\, d^2\mathbf{x}.$$

- However, this pairing is degenerate: for all $X \in \mathfrak{X}_{\mathrm{vol}}$,

$$\langle df, X \rangle = \int_{\mathcal{F}_0} df(X)\, d^2\mathbf{x} = \int_{\partial\mathcal{F}_0} f\, X \cdot \mathbf{n}\, dl - \int_{\mathcal{F}_0} f\, \nabla \cdot X\, d^2\mathbf{x} = 0.$$

- Hence, the “right” dual is given by

$$\mathfrak{X}^*_{\mathrm{vol}} = \frac{\Omega^1(\mathcal{F}_0)}{d\Omega^0(\mathcal{F}_0)} \cong d\Omega^1(\mathcal{F}_0),$$

where the last isomorphism is given by $[\alpha] \mapsto d\alpha$.

The momentum map associated to $\mathrm{Diff}_{\mathrm{vol}}$: vorticity

- The particle relabeling symmetry induces a momentum map

$$J : T(\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2)) \to \mathfrak{X}^*_{\mathrm{vol}}$$

given by (recall that $\mathbf{u} = \dot{\varphi} \circ \varphi^{-1}$)

$$J(\varphi, g; \dot{\varphi}, \dot{g}) = \mathrm{d}(\varphi^* \mathbf{u}^\flat).$$

- In classical notation, this becomes $J = \phi^*(\nabla \times \mathbf{u})$; *the vorticity of the fluid in the reference configuration is the momentum map associated to the the particle relabeling symmetry.*

Symplectic reduction with respect to $\mathrm{Diff}_{\mathrm{vol}}$: introducing point vortices

- Form the quotient $J^{-1}(\mu)/G_\mu$. If μ is a regular value, then this is a manifold with a reduced symplectic form Ω_μ characterized by

$$\pi_\mu^* \Omega_\mu = \iota_\mu^* \Omega.$$

- In the context of fluid dynamics, working on a level set $J^{-1}(\mu)$ means making an assumption about the vorticity of the system.
- Here: N point vortices of strength Γ_i, $i = 1, \ldots, N$:

$$\mu = \sum_{i=1}^{N} \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i) \mathrm{d}x \wedge \mathrm{d}y$$

- What does $J^{-1}(\mu)/G_\mu$ look like?

Special case: cotangent bundle reduction

- The reduced symplectic manifold is related to

$$J^{-1}(\mu)/G_\mu \rightsquigarrow T^*(Q/G) \times_{Q/G} Q/G_\mu;$$

  the "diffeomorphism" depends on the choice of $\mathcal{A}$.
- The reduced symplectic form is given by

$$\Omega_\mu = \Omega_{\mathrm{can}} - \beta_\mu$$

  where β_μ is a magnetic 2-form on Q/G_μ given by $\pi^*_{Q,G_\mu}\beta_\mu = \mathrm{d}\langle \mu, \mathcal{A}\rangle$.
- Using the Cartan structure formula

$$\mathrm{d}\mathcal{A} = \mathcal{B} - [\mathcal{A}, \mathcal{A}],$$

  we can relate β_μ to the curvature $\mathcal{B}$.

Cotangent bundle reduction: point vortices

- The isotropy group $\mathrm{Diff}_{\mathrm{vol},\mu}$ consists of all diffeomorphisms preserving μ: $\phi^*\mu = \mu$.
- For point vortices, $\phi \in \mathrm{Diff}_{\mathrm{vol},\mu}$ iff

$$\phi(\mathbf{x}_i) = \mathbf{x}_i \quad (\text{for } i = 1, \ldots, N).$$

- The quotient space Q/G_μ is in this case

$$(\mathrm{Emb}(\mathcal{F}_0, \mathbb{R}^2) \times SE(2))/\mathrm{Diff}_{\mathrm{vol},\mu} = \mathbb{R}^{2N} \times SE(2).$$

- The reduced cotangent bundle is hence

$$T^*(Q/G) \times_{Q/G} Q/G_\mu = T^*(SE(2)) \times \mathbb{R}^{2N};$$

product of a cotangent bundle and a co-adjoint orbit.

The reduced symplectic form has the vortex-rigid body interaction

The magnetic two-form β_μ is a two-form on $\mathbb{R}^{2N} \times \mathrm{SE}(2)$ and can be written as

$$\beta_\mu(\mathbf{x}_1, \ldots, \mathbf{x}_N; g)((\mathbf{v}_1, \ldots, \mathbf{v}_N, \dot{g}_1), (\mathbf{w}_1, \ldots, \mathbf{w}_N, \dot{g}_2)) \\ = \sum_{i=1}^{N} \Gamma_i \left(- d\mathbf{x}(\mathbf{v}_i, \mathbf{w}_i) + \mathrm{d}(\Psi_A \mathrm{d}\xi_A)(\dot{g}_1, \mathbf{v}_i; \dot{g}_2, \mathbf{w}_i) \right).$$

- Second term: encodes the vortex-rigid body interaction.
- $\sum_{i=1}^{N} \Gamma_i \,\mathrm{d}\mathbf{x}(\mathbf{v}_i, \mathbf{w}_i)$ is the Konstant-Kirillov-Souriau form on the co-adjoint orbit $\mathbb{R}^{2N}$.
- The stream functions Ψ_A are harmonic conjugates to Φ_A (elementary velocity potentials corresponding to the ξ_A-direction in $SE(2)$).

Reduction with respect to $SE(2)$ yields the BMR Poisson structure

The rigid body and point vortices are invariant under the action of $SE(2)$ on $T^*SE(2) \times \mathbb{R}^{2N}$. Factoring out this symmetry boils down to rewriting the equations of motion in body coordinates.

- Do Poisson reduction to reduce the magnetic symplectic structure on $T^*SE(2) \times \mathbb{R}^{2N}$ to the following Poisson structure on $\mathfrak{se}(2)^* \times \mathbb{R}^{2N}$:

$$\{f, k\}_{\mathrm{int}} = \frac{\delta f}{\delta \mu} \star \frac{\delta k}{\delta \mu} - \{f_{|P}, k_{|P}\}_{\mathbb{R}^{2N}} - \beta_\mu \left(\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu} \right) - \frac{\delta f}{\delta \mu} \lhd k + \frac{\delta k}{\delta \mu} \lhd f,$$

- In coordinates, this yields precisely the BMR Poisson structure.

The momentum map provides the link with the SMBK Poisson structure

Let $\xi^* \in \mathfrak{X}(SE(2) \times \mathbb{R}^{2N})$ be the infinitesimal generator corresponding to $\xi \in \mathfrak{se}(2)^*$. Define the $\mathcal{B}\mathfrak{g}$-potential $\phi : SE(2) \times \mathbb{R}^{2N} \to \mathfrak{se}(2)^*$ by

$$i_{\xi^*}\beta_\mu = \mathrm{d}\langle \phi, \xi \rangle.$$

- Looks like a momentum map, but for the magnetic two-form β_μ.
- Induces a shift map $\mathcal{S} : \mathfrak{se}(2)^* \times \mathbb{R}^{2N} \to \mathfrak{se}(2)^* \times \mathbb{R}^{2N}$, with

$$\mathcal{S}(\mu, \mathbf{x}) = (\mu - \phi(e, \mathbf{x}), \mathbf{x}).$$

- This is a Poisson map:

$$\{f \circ \mathcal{S}, g \circ \mathcal{S}\}_{\mathrm{BMR}} = \{f, g\}_{\mathrm{SMBK}} \circ \mathcal{S}.$$

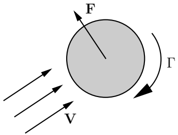
Rigid body with circulation

- Circulation: realized by having a point vortex at the center of mass.
- Symplectic reduction gives the Poisson bracket for a rigid body with circulation:

$$\{f,g\} = \{f,g\}_{\mathfrak{se}(2)^*} - \underbrace{\Gamma\left(\frac{\partial f}{\partial P_x}\frac{\partial g}{\partial P_y} - \frac{\partial g}{\partial P_x}\frac{\partial f}{\partial P_y}\right)}_{\text{curvature term}}.$$

- The resulting dynamics describes the motion of a rigid body under the influence of a *gyroscopic force due to the curvature term*. This is the familiar Kutta-Joukowski force, similar to the Lorentz force.

$$\begin{cases} \dot{\Pi} &= 0 \\ \dot{P}_x &= -\Pi P_y/\mathbb{I} - \Gamma P_y/m \\ \dot{P}_y &= \Pi P_x/\mathbb{I} + \Gamma P_x/m, \end{cases}$$



Conclusions

Geometric reduction yields the equations of motion for a rigid body interacting with point vortices.

- Symplectic reduction wrt to $\mathrm{Diff}_{\mathrm{vol}}$ brings in the vorticity;
- The momentum map associated to the residual $SE(2)$ action gives a shift map between BMR and SMBK.

Classical fluid dynamical effects have a geometric interpretation.

- The Kutta-Joukowski force;
- Analogy with magnetodynamics.

Open questions

This is probably only the tip of the iceberg...

- General distributions of vorticity: do reduction at a different value μ of J.
- Controllability of a fluid through rigid body motions. Stirring?
- Geometric integrators for fluid-solid interactions. Find a discrete analogue of this procedure of reduction by stages.
- Relative equilibria and their stability, chaos. For one vortex + body, the reduced phase space is a symplectic leaf $\mathcal{O}$ in $\mathfrak{se}(2)^* \times \mathbb{R}^2$, so $\dim \mathcal{O} = 4$. Independent commuting integrals: energy and rotations around symmetry axis.

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