AN INTRODUCTION TO RELATIVE EQUILIBRIA IN SYMMETRIC HAMILTONIAN SYSTEMS

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Abstract. These are the lecture notes of a mini-course delivered at the second meeting of young researchers of the Geometry, Mechanics and Control Network. The subject is an introduction to the theory of existence and stability for relative equilibria of Hamiltonian dynamical systems on symplectic manifolds invariant under a Hamiltonian action of a Lie group.

Contents

1. Introduction

Relative equilibria are the symmetric analogues of fixed equilibria for dynamical systems which are invariant under the action of a Lie group. Examples include circular orbits in the Kepler problem, rigid bodies and heavy tops rotating about an axis of symmetry, and rotating systems of point vortices with constant geometric configurations. The study of the existence of relative equilibria and their qualitative properties (stability, persistence, bifurcations, local dynamics around them, etc) is an important and active research topic, specially in the case of Hamiltonian systems, since relative equilibria of mechanical systems observable in Nature are those which are stable in a precise sense to be introduced in later. In these notes we will give an introduction to the most important aspects of the theory so that

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the reader will be able to access to the research literature. In particular, under some simplifying assumptions we will give a characterization of relative equilibria in any symmetric Hamiltonian system as well as sufficient conditions for their stability. However, many important topics will be left aside, as well as proofs of all the results. In particular, the important subjects of persistence an bifurcations of relative equilibria, relative equilibria for actions of non-compact symmetry groups, or relative equilibria lying in singular level sets of the momentum map will not be covered.

In these notes, M will denote a smooth manifold, $C^\infty(M)$ and $\mathfrak{X}(M)$ the sets of smooth functions and smooth vector fields on M respectively. G will denote a **compact connected** Lie group acting on M in a **free** way, we will write $g \cdot x$ for the action of $g \in G$ on $x \in M$. The Lie algebra of G and its dual will be denoted by $\mathfrak{g}$ and $\mathfrak{g}^*$. The fundamental vector fields of this action are $\xi_M \in \mathfrak{X}(M)$, for every $\xi \in \mathfrak{g}$, where

$$\xi_M(x) = \frac{d}{dt}\Big|_{t=0} e^{t\xi} \cdot x$$

where $e : \mathfrak{g} \to G$ is the exponential map for G.

2. Relative equilibria of vector fields

For $\epsilon \subset \mathbb{R}$, let $X \in \mathfrak{X}(M)$ define a dynamical system $\phi_X : \epsilon \times M \to M$ by the differential equation

$$\dot{\phi}_X(t,x) = X(\phi_X(t,x)). \tag{1}$$

That is, ϕ_X is the flow of X.

A point $x \in M$ is an **equilibrium** of ϕ_X if $\phi(t,x) = x$ for every $t \in \epsilon$. Equivalently, using (1), x is a critical point of the vector field X, that is

$$X(x) = 0.$$

Now suppose that an action of G on M is given and that X is invariant for this action. This is the same as requiring its flow to be equivariant, that is

$$\phi(t, g\cdot x) = g \cdot \phi(t,x)$$

for every $g \in G$, $t \in \epsilon$ and $x \in M$ for which the above equation is well defined.

Definition 1. *A solution $x(t) = \phi_X(t,x)$ of the dynamical system* (1) *is a* **relative equilibrium** *if there exists an element $\xi \in \mathfrak{g}$ such that*

$$x(t) = \phi_X(t,x) = e^{t\xi} \cdot x.$$

Remark 1.
1. This condition means that the evolution of x coincides with the group orbit of a one-parameter subgroup of G generated by $\xi \in \mathfrak{g}$.
2. In this situation, it is customary to say that x is a relative equilibrium of the vector field X. It is to be noted however, that if x is also a relative equilibrium, then every other point in its group orbit $G \cdot x$ is also a relative equilibrium.
3. The Lie algebra element ξ is called the **velocity** of the relative equilibrium.
4. As a particular case, an equilibrium of X is also a relative equilibrium with zero velocity.

Even in this simple setting, the relationship between equilibria and relative equilibria is better visualized using the process of symmetry reduction. Let M/G be the orbit space of the G-action and $\pi : M \to M/G$ the corresponding orbit map. Elements of M/G are denoted by $[x]$ where x is a representative in M, that is, $\pi(x) = [x]$.

By invariance of X and equivariance of ϕ_X we have the following properties:

(i) X is π-projectable, i.e. there exists a vector field $\overline{X} \in \mathfrak{X}(M/G)$ such that
$$\overline{X} = \pi_* X.$$

(ii) The flow of X is π-projectable and its projection is the flow of $\overline{X}$, i.e.
$$\pi\left(\phi_X(t,x)\right) = \phi_{\overline{X}}(t,[x])$$
when the above equation is well defined.

In view of these properties, it follows

Proposition 1. *Let $x \in M$ and $X \in \mathfrak{X}(M)$. Then the following are equivalent*

(i) *x is a relative equilibrium with velocity $\xi \in \mathfrak{g}$.*
(ii) *$X(x) = \xi_M(x)$.*
(iii) *$[x] \in M/G$ is an equilibrium for the dynamical system induced by $\overline{X} \in \mathfrak{X}(M/G)$, that is,*
$$\overline{X}([x]) = 0.$$

*The dynamics on M/G associated to $\overline{X}$ is called the **reduced dynamics** of X.*

Example: Let $M = \mathbb{R}^3 \backslash \{0\}$ and $X(x,y,z) = (-(x^2+y^2)y + zx, (x^2+y^2)x + zy, z^2)$. The vector field X is invariant under the action of SO(2), the group of rotations around the z-axis. The associated dynamical system is given by
$$\begin{aligned} \dot{x} &= -(x^2+y^2)y + zx \\ \dot{y} &= (x^2+y^2)x + zy \\ \dot{z} &= z^2 \end{aligned}$$

In view of Proposition 1, it is not necessary to find the explicit dynamics of the system (the solution to the above system of ODE's) in order to find all of its relative equilibria. We start by computing the fundamental fields of the (SO(2)-action. It is straightforward to get
$$\xi_M(\mathbf{x}) = \xi \mathbf{e_3} \times \mathbf{x}, \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3,\ \xi \in \mathbb{R} \simeq \mathfrak{so}(2),\ \mathbf{e_3} = (0,0,1),$$
that is, $\xi_M(x,y,z) = (-\xi y, \xi x, 0)$.

Therefore, $\mathbf{x}$ will be a relative equilibrium with velocity ξ if and only if
$$\begin{aligned} -\xi y &= -(x^2+y^2)y + zx, \\ \xi x &= (x^2+y^2)x + zy, \\ 0 &= z^2. \end{aligned}$$

From here, we obtain immediately

(i) There are no (fixed) equilibria.
(ii) There are no relative equilibria outside the xy-plane,
(iii) Every point in the xy-plane is a relative equilibrium with velocity $\xi = x^2 + y^2 > 0$. Its dynamical evolution coincides with the orbit of SO(2) at $(x,y,0)$ (in this case the one-parameter subgroup of G generated by ξ coincides with G since SO(2) is one dimensional. This means that the dynamical orbit of $(x,y,0)$ describes a circle in the plane $z = 0$ of radius

$\sqrt{x^2+y^2}$ centered at the origin with constant velocity x^2+y^2 pointing in the direction of $\mathbf{e_3}$. That is, the resulting circle has positive orientation

(iv) There are no relative equilibria with negative velocity (circles with negative orientation).

3. Symmetric Hamiltonian systems

We now introduce additional structure in our manifold M in order to define more interesting dynamical systems. Let ω be a symplectic form in M and $h \in C^\infty(M)$ a **Hamiltonian**. With this data we can construct a dynamical system associated to the vector field X_h defined by

$$\omega(X_h, \cdot) = \mathbf{d}h. \tag{2}$$

In other words, X_h is the symplectic gradient of h. It is called the **Hamiltonian vector field** corresponding to h. The associated dynamical system, or equivalently, the triple (M, ω, h) is called a **Hamiltonian system**.

Equilibria: From (2), it is clear that $x \in M$ is an equilibrium of the Hamiltonian system (M, ω, h) if and only if $\mathbf{d}h(x) = 0$. That is, ***the equilibria of a Hamiltonian system are the critical points of the Hamiltonian.***

Characterization of equilibria as critical points of a function is a very useful method that can be implemented also for relative equilibria in the symmetric context. But for that, it will be necessary to define a natural class of Lie group actions for Hamiltonian dynamical systems.

Hamiltonian actions, momentum maps: An action of G on the symplectic manifold (M, ω) is called Hamiltonian if

(i) G acts by symplectomorphisms.

(ii) There exists a map $\mathbf{J} : M \to \mathfrak{g}^*$ such that for all $\xi \in \mathfrak{g}$ and $x \in M$

$$\omega(\xi_M(x), \cdot) = \mathbf{d}\langle \mathbf{J}(x), \xi \rangle.$$

That is, every fundamental vector field is a Hamiltonian vector field. The map $\mathbf{J}$ is called the ***momentum map*** for the action.

(iii) For all $g \in G$ and $x \in M$

$$\mathbf{J}(g \cdot x) = \mathrm{Ad}^{*\,-1}_g \mathbf{J}.$$

That is, the momentum map is equivariant.

Given a G-invariant Hamiltonian function, where G acts in a Hamiltonian fashion on (M, ω), the quadruple $(M, \omega, G, \mathbf{J}, h)$ defines a symmetric Hamiltonian system, since the Hamiltonian vector field X_h is G-invariant. A fundamental distinguished property of symmetric Hamiltonian systems that will be on central importance is the following result.

Theorem 1 (Noether's Theorem)**.** *The dynamics of a symmetric Hamiltonian system $(M, \omega, G, \mathbf{J}, h)$ leaves invariant the fibers of $\mathbf{J}$. In other words, the components of the momentum map are constants of motion (conserved quantities).*

A familiar example of a momentum map is provided by $SO(3)$ Hamiltonian actions, for which $\mathbf{J}$ is the usual angular momentum, that is conserved for $SO(3)$-invariant Hamiltonian systems, as a realization of Noether's theorem. A more subtle example of a momentum maps is the divergence of electric fields, where the symplectic manifold is the cotangent bundle of the space of vector potentials on $\mathbb{R}^3$

and the Hamiltonian action is obtained by lifting the group of gauge transformations of potentials by functions on $\mathbb{R}^3$.

4. Symplectic and Poisson reduction

In Proposition 1, (ii) we have seen that relative equilibria for the dynamics of a G-invariant vector field on M correspond to fixed equilibria of the reduced dynamics on M/G. In the framework of Hamiltonian actions on symplectic manifolds there are two natural orbit spaces that we can consider.

Proposition 2. *Let G act in a Hamiltonian fashion on the symplectic manifold (M, ω) with associated equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Let $\{\cdot, \cdot\}$ be the Poisson bracket on M induced by ω. Then we have:*

(i) ***Poisson reduction.*** *The orbit space M/G is a Poisson manifold endowed with the reduced Poisson bracket defined by*

$$\{f, g\}_{red}([x]) = \{f \circ \pi, g \circ \pi\}(x),$$

where $f, g \in C^\infty(M/G)$ and $\pi : M \to M/G$ is the orbit map for the action of G on M.

(ii) ***Symplectic reduction.*** *Let $\mu \in \mathfrak{g}^*$. Then $\mathbf{J}^{-1}(\mu)$ is G_μ-invariant submanifold of M, and the orbit space $\mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold with a reduced symplectic form defined by*

$$\pi_\mu^* \omega_\mu = \iota_\mu^* \omega.$$

Here G_μ is the stabilizer of μ with respect to the coadjoint representation of G, $\iota_\mu : \mathbf{J}^{-1}(\mu) \to M$ the natural inclusion, and $\pi_\mu : \mathbf{J}^{-1}(\mu) : \mathbf{J}^{-1}(\mu)/G$ the orbit map for the action of G_μ on $\mathbf{J}^{-1}(\mu)$.

The momentum map induces a natural inclusion of $\mathbf{J}^{-1}(\mu)/G$ in M/G. With respect to this, $(\mathbf{J}^{-1}(\mu)/G_\mu, \omega_\mu)$ is a symplectic leaf of the Poisson manifold $(M/G, \{\cdot, \cdot\}_{\mathrm{red}})$.

These two reduced spaces play an important role in the description of the dynamics of symmetric Hamiltonian systems on M in the following way: If h is a G-invariant Hamiltonian on M, it drops to a function $\overline{h}$ on M/G. In the same way, the G-invariant Hamiltonian vector field X_h drops to a vector field $\overline{X_h}$ on M/G. It is a consequence of reduction that $\overline{X_h}$ is a Poisson Hamiltonian vector field on M/G with Hamiltonian $\overline{h}$, in the sense that

$$\overline{X_h} = \{\cdot, \overline{h}\}_{\mathrm{red}}.$$

From the symplectic point of view, the restriction h^μ of h to $\mathbf{J}^{-1}(\mu)$ is G_μ-invariant, and then there is an induced function $\overline{h^\mu}$ on $\mathbf{J}^{-1}(\mu)/G_\mu$. Also, by Noether's Theorem, X_h is tangent to $\mathbf{J}^{-1}(\mu)$ and G_μ-invariant, so it projects to a vector field $\overline{X_h^\mu}$ on the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$. Again by reduction, $\overline{X_h^\mu}$ is a Hamiltonian vector field with Hamiltonian $\overline{h^\mu}$ on the symplectic reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ according to

$$\omega_\mu(\overline{X_h^\mu}, \cdot) = \mathbf{d}\overline{h^\mu}.$$

Of course, the reduced Hamiltonian system induced by $\overline{h}$ on M/G must preserve its symplectic leaves, and it is easy to see that the restriction of this Poisson Hamiltonian system to a leaf $\mathbf{J}^{-1}(\mu)/G$ gives precisely the Hamiltonian system corresponding to $\overline{h^\mu}$.

5. Hamiltonian relative equilibria: characterization

As for any vector field, we can use Definition 1 to locate its relative equilibria with their velocities. However, in the Hamiltonian setting, due to the presence of extra structure it is possible to refine the characterization of relative equilibria given in Proposition 1.

Theorem 2. *Let $(M, \omega, G, \mathbf{J}, h)$ be a symmetric Hamiltonian system, and x a point in M with momentum $\mu = \mathbf{J}(x)$. The following are equivalent.*

(i) *x is a relative equilibrium,*
(ii) *there exists a Lie algebra element $\xi \in \mathfrak{g}$ such that $X_h(x) = \xi_M(x)$,*
(iii) *$[x] \in M/G$ is a fixed equilibrium for the reduced Poisson Hamiltonian system $(M/G, \{\cdot,\cdot\}_{red}, \overline{h})$,*
(iv) *$[x] \in \mathbf{J}^{-1}(\mu)/G$ is a fixed equilibrium for the reduced Hamiltonian system $(\mathbf{J}^{-1}(\mu)/G, \omega_\mu, \overline{h^\mu})$,*
(v) *there exists a Lie algebra element $\xi \in \mathfrak{g}$ such that*

$$\mathbf{d}h_\xi(x) = 0, \tag{3}$$

where $h_\xi(x) := h(x) - \langle \mathbf{J}(x), \xi \rangle)(x)$.

Remark 2.
1. The Lie algebra element ξ in (ii) and (v) is called the velocity of the relative equilibrium.
2. The function h_ξ in (v) is called the augmented Hamiltonian. From (3), it follows that ***Hamiltonian relative equilibria are exactly the critical points of the augmented Hamiltonian***, just as fixed equilibria were previously found to be critical points of the Hamiltonian alone.
3. Equation (3) can be naturally interpreted as the condition for x to be a critical point of h constrained to the invariant submanifold $\mathbf{J}^{-1}(\mu)$. Then, the velocity of the relative equilibrium corresponds to a Lagrange multiplier for the constrained problem.
4. It is straightforward to verify, using the equivariance of $\mathbf{J}$ and Noether's Theorem that $\mathrm{ad}_\xi \mu = 0$. That is, ξ belongs to $\mathfrak{g}_\mu$, the Lie algebra of the stabilizer G_μ.

In concrete applications, one usually tests condition (v) with different choices of ξ in order to find the points x that are relative equilibria with that given velocity.

6. Liapunov stability, orbital stability and stability modulo a subgroup

The second natural step in the qualitative investigation of equilibria and relative equilibria of vector fields is the study of their stability. There are some standard notions of stability in the literature. We will review their definitions for (symmetric) Hamiltonian systems, although they are valid for flows of general (G-)invariant vector fields. We will continue in the setup of previous section, considering a symmetric Hamiltonian system $(M, \omega, G, \mathbf{J}, h)$.

Definition 2. *Let x be a fixed equilibria. Then x it is said to be* ***Liapunov stable*** *if for every neighborhood $U \ni x$, there exists a neighborhood $O \ni x$ such that*

$$\phi_{X_h}(t, O) \subseteq U$$

for all t.

Definition 3. *Let $x(t)$ be the integral curve of X_h with initial condition x. Then x is said to be* ***orbitally stable*** *if for every neighborhood $U \ni x(t)$ there is a neighborhood $O \ni x$ such that,*

$$\phi_{X_h}(t, O) \subseteq U$$

for all t.

It was observed that the notion of orbital stability is too restrictive for being considered as the right definition of stability for Hamiltonian relative equilibria. The reason is that, in the presence of symmetry, relative equilibria which could be physically considered as stable fail to be orbital stable due to the presence of a drift along the group directions. The right notion of stability of relative equilibria in the Hamiltonian situation is that of stability modulo a subgroup. We will see in the next section that this has a natural interpretation in terms of Liapunov stability of the corresponding equilibrium for the reduced dynamics.

Definition 4. *Let x be a relative equilibrium and $A \subseteq G$. Then, x is said to be* ***stable modulo*** *A if for every A-invariant neighborhood $U \ni x(t)$ there is a neighborhood $O \ni x$ such that*

$$\phi_{X_h}(t, O) \subseteq U$$

for all t.

In our setup, mostly because G is compact, the right notion of stability for Hamiltonian stability is stability modulo G_μ (also referred to as G_μ-stability) where $\mu = \mathbf{J}(x)$, i.e. making $A = G_\mu$ in Definition 4. If the compactness assumption of G is dropped, one may have to consider more general subsets A satisfying $G_\mu \subset A \subseteq G$.

Important Remark. To a great extent, all the results stated until here remain true (or require mild modifications) for symmetric Hamiltonian systems not satisfying our starting hypotheses (G compact and connected acting freely on M). However from this point these hypotheses become essential, and most of the results of the subsequent sections require major modifications in order to accommodate group actions failing to satisfy one or some of these hypotheses.

7. Dynamical interpretation of the stability on a reduced phase space

Since Hamiltonian relative equilibria with momentum μ correspond to fixed equilibria for the reduced Hamiltonian system on the corresponding symplectic reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$, it is natural to ask if G_μ-stability of a relative equilibrium is equivalent to Liapunov stability of the reduced equilibrium. The following important result states this equivalence, and it shows why G_μ-stability is the natural interpretation of nonlinear stability for Hamiltonian relative equilibria.

Theorem 3. *Let $(M, \omega, G, \mathbf{J}, h)$ be a symmetric Hamiltonian system and $x \in M$ a relative equilibrium with momentum $\mu = \mathbf{J}(x)$. Then x is G_μ-stable if and only if the reduced equilibrium $[x]$ is Liapunov stable in the symplectic reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$.*

It is important to notice that this is a highly nontrivial result, since at a first sight it would seem that lifting the property of Liapunov stability on the reduced space to the original manifold M can only guarantee G_μ-stability of x *inside* $\mathbf{J}^{-1}(\mu)$. Or in other words, in principle all that one could conclude is that x is G_μ-stable for U (as in Definition 2) belonging to $\mathbf{J}^{-1}(\mu)$, not generally to M. The power of Theorem 3 is that it guarantees that this remains true for arbitrary G_μ-invariant neighborhoods. This result is usually stated as "stability for momentum-preserving perturbations implies stability for arbitrary perturbations". This implication depends crucially on the compactness of G, although this could be relaxed to require that the Lie algebra $\mathfrak{g}$ admits a Ad_{G_μ}-invariant inner product, in which case the relative equilibrium is said to have a ***split*** momentum value.

8. The symplectic slice

The symplectic slice at a point x with momentum $\mu = \mathbf{J}(x)$ of a symplectic manifold endowed with a Hamiltonian action is a maximal symplectic subspace of the symplectic orthogonal to the tangent space to the orbit at x. This is a fundamental tool in the study of Hamiltonian actions, since it encodes the local symplectic geometry of M near the group orbit through x via the Marle-Guillemin-Sternberg symplectic tubular neighborhood. Additionally, the symplectic slice is isomorphic to the tangent space to $\mathbf{J}^{-1}(\mu)/G_\mu$ at $[x]$. Therefore, it provides a way of realizing (non-uniquely) the tangent space to the reduced symplectic space as a vector subspace of the tangent space to the unreduced space. This last property will be the most interesting one for us since, as we shall see in the next section, its combination with Theorem 3 will produce a method to test G_μ-stability of Hamiltonian relative equilibria.

Definition 5. *Let G act in a Hamiltonian fashion on the symplectic manifold (M,ω) with equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Let $x \in M$ with momentum $\mathbf{J}(x) = \mu$ and denote by $\mathfrak{g}$ and $\mathfrak{g}_\mu$ the Lie algebras of G and G_μ respectively.*

Let $N \in T_xM$ be defined by the identity

$$\ker T_x\mathbf{J} = \mathfrak{g}_\mu \cdot x \oplus N.$$

*Then any such N is called a **symplectic slice** at x.*

Remark 3. The definition of the symplectic slice may seem unrelated to the the properties stated at the beginning of the section. However, using the definition of the momentum map and its equivariance, it is easy to see that

$$\ker T_x\mathbf{J} = (\mathfrak{g} \cdot x)^\omega \quad \text{and} \quad \mathfrak{g}_\mu \cdot x = \mathfrak{g} \cdot x \cap (\mathfrak{g} \cdot x)^\omega.$$

It is now an exercise in linear symplectic algebra to see that N defined as

$$(\mathfrak{g} \cdot x)^\omega = N \oplus \mathfrak{g} \cdot x \cap (\mathfrak{g} \cdot x)^\omega$$

is a maximal symplectic subspace of $(\mathfrak{g} \cdot x)^\omega$.

For the second property, note that N is isomorphic to $\ker T_x\mathbf{J}/\mathfrak{g}_\mu \cdot x$, and that $\ker T_x\mathbf{J} = T_x\mathbf{J}^{-1}(\mu)$. Therefore N is the linearization of the symplectic reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ at $[x]$. That is, N is isomorphic to $T_{[x]}\left(\mathbf{J}^{-1}(\mu)/G_\mu\right)$.

9. The Energy-Momentum method

The Energy-Momentum method is the commonly used approach to testing G_μ-stability of Hamiltonian relative equilibria. Morally, it consists in testing the Liapunov stability of the corresponding fixed equilibrium on the reduced symplectic space, but this test is performed on the unreduced space. This method builds critically on Theorem 3 and Definition 5, in particular in the isomorphism of N and $T_{[x]}\left(\mathbf{J}^{-1}(\mu)/G_\mu\right)$.

Let x be a relative equilibrium with momentum μ and velocity ξ. The key observations leading to the Energy-Momentum method are the following: First, by Theorem 2, x is a critical point for the augmented Hamiltonian h_ξ, therefore the Hessian $\mathbf{d}^2_x h\xi$ is well defined. Second, again by Theorem 2 the reduced equilibrium $[x]$ is a critical point for $\overline{h^\mu}$, and therefore its Hessian $\mathbf{d}^2_{[x]}\overline{h^\mu}$ is also well-defined. Third, recall that N is a vector subspace of T_xM, and let $\mathbf{d}^2_x h_\xi\big|_N$ be the symmetric bilinear form on N obtained by restricting the Hessian of h_ξ at x to N. Let $T_x\pi_\mu : N \to T_{[x]}\left(\mathbf{J}^{-1}(\mu)/G_\mu\right)$ be the corresponding linear isomorphism. It can be proved that

$$(T_x\pi_\mu)^*\mathbf{d}^2_{([x])}\overline{h^\mu} = \mathbf{d}^2_x h_\xi\big|_N. \tag{4}$$

Now, since $\overline{h^\mu}$ is constant on the integral curves of the reduced Hamiltonian dynamics, it is a Liapunov function that can be used to obtain sufficient conditions for the Liapunov stability of $[x]$. Then, from the standard theory of dynamical systems, it follows that such a necessary condition is given by the definiteness of $[x]$, or in other words, if the bilinear form $\mathbf{d}^2_{[x]}\overline{h^\mu}$ is strictly positive (or negative) definite. This condition, equated with (4) provides sufficient conditions for the G_μ-stability of x which can be tested on the original unreduced space. This is known as the Energy-Momentum method, and it is summarized in the following theorem.

Theorem 4 (Energy-Momentum method.)**.** *Let $(M, \omega, G, \mathbf{J}, h)$ be a symmetric Hamiltonian system, and $x \in M$ a relative equilibrium with momentum $\mu = \mathbf{J}(x)$ and velocity ξ. Then the augmented Hamiltonian $h_\xi = h - \langle \mathbf{J}(\cdot), \xi\rangle$ has a critical point at x. If the bilinear form*

$$\mathbf{d}^2_x h_\xi\big|_N$$

is definite for some (and hence any) linear complement N of $\mathfrak{g}_\mu \cdot x$ in $\ker T_x\mathbf{J}$, then the relative equilibrium x is G_μ-stable.

10. An example: The spherical pendulum

The spherical pendulum is a symmetric Hamiltonian system constructed over the configuration space S^2. The ingredients needed to implement its Hamiltonian structure and study its relative equilibria are:

(i) The symplectic manifold $M = T^*S^2$, equipped with its canonical cotangent bundle symplectic structure. Using the natural embedding of S^2 in $\mathbb{R}^3$ and the Euclidean inner product on $\mathbb{R}^3$ we can characterize T^*S^2 as

$$T^*S^2 = \{(\mathbf{x}, \mathbf{p}) \,:\, \|\mathbf{x}\| = 1,\, \mathbf{x}\cdot\mathbf{p} = 0\}.$$

(ii) The Hamiltonian of the spherical pendulum is given by

$$h(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\|\mathbf{p}\|^2 + mgl\mathbf{x}\cdot\mathbf{e_3},$$

where m is the mass of the pendulum, l its length (the radius of S^2), and g is the gravitational constant. In this realization, $\mathbf{x}$ is the position vector of the pendulum bob in $\mathbb{R}^3$ and $\mathbf{p}$ is its linear momentum.

(iii) The Lie group G leaving invariant this Hamiltonian is S^1, rotations around $\mathbf{e_3}$. The action is given by

$$(\varphi, (\mathbf{x}, \mathbf{p})) \mapsto (R_\varphi \mathbf{x}, R_\varphi \mathbf{p}).$$

It is easy to see that this action is Hamiltonian. In order to work with a free action, from now on we will omit the points of the form $(\pm \mathbf{e_3}, \mathbf{p})$. This corresponds to consider as our symplectic manifold the cotangent bundle of the 2-sphere with the north and south poles removed.

(iv) The Lie algebra of S^1 is isomorphic to $\mathbb{R}$, as well as its dual $\mathfrak{g}^*$. Let $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, both spaces identified with $\mathbb{R}$. Then the natural pairing is given by the expression $\langle \mu, \xi \rangle = \mu\xi$ (as the multiplication of real numbers). The momentum map for this action is easily found to be

$$\mathbf{J}(\mathbf{x}, \mathbf{p}) = (\mathbf{x} \times \mathbf{p}) \cdot \mathbf{e_3}.$$

Existence criterion. In order to locate relative equilibria for the spherical pendulum we construct the augmented potential. From the above definitions, it follows that

$$h_\xi(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}\|\mathbf{p}\|^2 + mgl\mathbf{x} \cdot \mathbf{e_3} - \xi(\mathbf{x} \times \mathbf{p}) \cdot \mathbf{e_3}.$$

Relative equilibria of h are critical points of h_ξ, therefore we compute

$$\mathbf{d}_{(\mathbf{x},\mathbf{p})} h_\xi(\dot{\mathbf{x}}, \dot{\mathbf{p}}) = (mgl\mathbf{e_3} - \xi \mathbf{p} \times \mathbf{e_3}) \cdot \dot{\mathbf{x}} + \left(\frac{\mathbf{p}}{m} - \xi \mathbf{e_3} \times \mathbf{x}\right) \cdot \dot{\mathbf{p}}. \tag{5}$$

Note that

$$T_{(\mathbf{x},\mathbf{p})}(T^*S^2) = \{(\dot{\mathbf{x}}, \dot{\mathbf{p}}) \,:\, \dot{\mathbf{x}} \cdot \mathbf{x} = 0,\, \dot{\mathbf{x}} \cdot \mathbf{p} + \mathbf{x} \cdot \dot{\mathbf{p}} = 0\}.$$

Solving for variations of the form $(\dot{\mathbf{x}}, \dot{\mathbf{p}}) = (\mathbf{0}, \dot{\mathbf{p}})$, we get

$$\mathbf{p} = m\xi \mathbf{e_3} \times \mathbf{x}. \tag{6}$$

If we introduce this condition in(5) we get that a point $(\mathbf{x}, \mathbf{p})$ is a relative equilibrium if and only if (6) holds simultaneously with

$$(mgl\mathbf{e_3} - \xi(m\xi \mathbf{e_3} \times \mathbf{x}) \times \mathbf{e_3}) \cdot \dot{\mathbf{x}} = 0 \quad \text{for all} \quad \dot{\mathbf{x}} \in T_{\mathbf{x}} S^2. \tag{7}$$

Manipulating (7) and using the standard identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$, as well as $\mathbf{x} \cdot \dot{\mathbf{x}} = 0$ for $\dot{\mathbf{x}} \in T_{\mathbf{x}} S^2$, it follows that relative equilibria with velocity ξ for the spherical pendulum are all the pairs $(\mathbf{x}, \mathbf{p})$ satisfying

$$\mathbf{p} = m\xi \mathbf{e_3} \times \mathbf{x}, \tag{8}$$

$$\xi^2 = \tfrac{-gl}{\mathbf{x} \cdot \mathbf{e_3}}. \tag{9}$$

One can see immediately that (9) only has solutions if the pendulum bob is in the lower hemisphere (excluding the equator). Moreover, each point in the lower hemisphere is a relative equilibrium for exactly one choice of $|\xi|$. From (8) it follows that the evolution of the relative equilibrium is then along the circle containing $\mathbf{x}$ parallel to the $\langle \mathbf{e_1}, \mathbf{e_2} \rangle$ plane, with constant angular velocity $\pm|\xi|$.

Stability criterion. We now apply the reduced energy-momentum method to the study of the stability of these relative equilibria. It is straightforward to obtain, by differentiation (5), that

$$\langle(\mathbf{a_1},\mathbf{b_1}),\mathbf{d}^2_{(\mathbf{x},\mathbf{p})}h_\xi\cdot(\mathbf{a_2},\mathbf{b_2})\rangle=-\xi\mathbf{e_3}\cdot(\mathbf{a_1}\times\mathbf{b_2}+\mathbf{a_2}\times\mathbf{b_1})+\frac{1}{m}\mathbf{b_1}\cdot\mathbf{b_2} \tag{10}$$

The next step is to compute G_μ. It is actually not necessary to know the value of μ since, in this example, the symmetry group G is abelian, hence the coadjoint representation is trivial and therefore $G_\mu = G$. Consequently

$$\mathfrak{g}_\mu\cdot(\mathbf{x},\mathbf{p})=\mathfrak{g}\cdot(\mathbf{x},\mathbf{p})=\{\lambda(\mathbf{e_3}\times\mathbf{x},\mathbf{e_3}\times\mathbf{p})\,:\,\lambda\in\mathbb{R}\}. \tag{11}$$

Using (8), at the relative equilibrium

$$\mathfrak{g}_\mu\cdot(\mathbf{x},\mathbf{p})=\{\lambda(\mathbf{e_3}\times\mathbf{x},m\xi(\mathbf{x}\cdot\mathbf{e_3})\mathbf{e_3}-m\xi\mathbf{x})\,:\,\lambda\in\mathbb{R}\}. \tag{12}$$

In order to compute the kernel of the derivative of the momentum map, we differentiate $\mathbf{J}$ to obtain

$$T_{(\mathbf{x},\mathbf{p})}\mathbf{J}\cdot(\dot{\mathbf{x}},\dot{\mathbf{p}})=(\mathbf{p}\times\mathbf{e_3})\cdot\dot{\mathbf{x}}-(\mathbf{x}\times\mathbf{e_3})\cdot\dot{\mathbf{p}}.$$

Substituting (8) and using $\dot{\mathbf{x}}\in T_{\mathbf{x}}S^2$ we obtain

$$\ker T_{(\mathbf{x},\mathbf{p})}\mathbf{J}=\{(\dot{\mathbf{x}},\dot{\mathbf{p}})\,:\,\dot{\mathbf{x}}\cdot\mathbf{e_3}=\frac{(\mathbf{e_3}\times\mathbf{x})\cdot\dot{\mathbf{p}}}{m\xi\mathbf{x}\cdot\mathbf{e_3}},\,\dot{\mathbf{x}}\cdot\mathbf{x}=0,\,m\xi\dot{\mathbf{x}}\cdot(\mathbf{e_3}\times\mathbf{x})+\mathbf{x}\cdot\dot{\mathbf{p}}=0\}.$$

If we use the basis $\{\mathbf{e_3},\mathbf{x},\mathbf{e_3}\times\mathbf{x}\}$ for $\mathbb{R}^3$, it is not difficult to show that a complement N to $\mathfrak{g}_\mu\cdot(\mathbf{x},\mathbf{p})$ in $\ker T_{(\mathbf{x},\mathbf{p})}\mathbf{J}$ is given by all vectors of the form $(\dot{\mathbf{x}},\dot{\mathbf{p}})$ satisfying

$$\begin{aligned}\dot{\mathbf{x}}&=-\frac{v}{m\xi\cos\theta}\,\mathbf{e_3}-\frac{v}{m\xi}\,\mathbf{x}+m\xi u\,\mathbf{e_3}\times\mathbf{x}\\ \dot{\mathbf{p}}&=\frac{(1+m^2\xi^2\sin^2\theta)u}{\cos\theta}\,\mathbf{e_3}+u\,\mathbf{x}+v\,\mathbf{e_3}\times\mathbf{x}\end{aligned}$$

with $u,v\in\mathbb{R}$, and where we have introduced the angle θ as $\cos\theta=-\mathbf{x}\cdot\mathbf{e_3}$ (θ is the angle between the pendulum and the negative $\mathbf{e_3}$ axis). This means that the vector space N is two dimensional and the above equations parametrize N by the coordinates (u,v).

If we define $A=\frac{(1+m^2\xi^2\sin^2\theta)}{\cos\theta}$ and we introduce the above parametrization of N in (10) we obtain

$$\begin{aligned}\langle(\mathbf{a_1},\mathbf{b_1}),\mathbf{d}^2_{(\mathbf{x},\mathbf{p})}h_\xi\big|_N\cdot(\mathbf{a_2},\mathbf{b_2})\rangle&=\frac{1}{m}\left[(A^2-2A\cos\theta+2m^2\xi^2\sin^2\theta+1)\,u_1u_2\right.\\&\quad\left.+3\sin^2\theta\,v_1v_2\right].\end{aligned}$$

The eigenvalues of this bilinear form are obviously $\frac{A^2-2A\cos\theta+2m^2\xi^2\sin^2\theta+1}{m}$ and $\frac{3\sin^2\theta}{m}$. The second of them is positive since the south pole of S^2 has been removed. Therefore, the Energy-Momentum method says that the spherical pendulum at this relative equilibrium $(\mathbf{x},\mathbf{p})$ with $\mathbf{p}=m\xi\,\mathbf{e_3}\times\mathbf{x}$ and velocity ξ satisfying the condition $\xi^2\cos\theta=gl$ is stable if

$$A^2-2A\cos\theta+2m^2\xi^2\sin^2\theta+1>0.$$

But this expression is obviously positive for any θ. Therefore all the relative equilibria of the spherical pendulum are stable.

Remark 4. The spherical pendulum provides one of the simplest not trivial examples of the applicability of the Energy-Momentum method. In more complicated situations one finds that the Energy-Momentum method can distinguish by their stability some of the relative equilibria found in a symmetric Hamiltonian system. For instance, an axisymmetric heavy top in upright position has relative equilibria consisting in the top rotating with constant angular velocity around the common axis of symmetry of the body and the direction of gravity. These relative equilibria, known as sleeping Lagrange tops, exist for any angular velocity. However the implementation of the Energy-Momentum method for this system shows that for those which are stable the modulus of the angular velocity must be greater than a certain lower bound (the classical fast top condition).

11. Comments and further directions

As it was already said, most of the results exposed in this notes remains valid if non-compact groups or non-free actions are considered. This holds in particular for the existence theory, but it is definitely not true for the stability theory. The topics covered in these notes are further expanded in [10], an accessible textbook that contains a detailed exposition of the theory of Hamiltonian relative equilibria under the hypotheses assumed here, including several worked examples. The definition of G_μ-stability, together with its dynamical interpretation, and most of the results of sections 6, 7, 8 and 9 appeared originally in [17]. The theory of reduction sketched in section 4 has its origins in the foundational paper [11]. A detailed exposition of the reduction theory of Hamiltonian actions can be found in [16]. As for examples, the Energy-Momentum method has been successfully applied to the study of a variety of complicated Hamiltonian systems. Following my personal taste, I find specially interesting its application to the heavy top ([7]), pseudo-rigid bodies ([9]), molecules ([13]) and systems of point vortices ([6]).

A relevant sophistication of the Energy-Momentum method exists for an important family of Hamiltonian systems, the so-called simple mechanical systems. These are systems for which the symplectic manifold is the cotangent bundle of a Riemannian manifold, the group action is the lift of an action by isometries on the base, and the Hamiltonian is of the form "kinetic plus potential" energy, where the kinetic energy is constructed with the Riemannian metric and the potential energy is the lift of an invariant function on the base. For these systems, the ***reduced Energy-Momentum method*** is a particular algorithm to implement the Energy-Momentum method such that the bilinear form that one needs to test to conclude stability is automatically put into block-diagonal form, therefore simplifying enormously the computations involved. For instance the spherical pendulum is a simple mechanical system. The implementation of the reduced Energy-Momentum method would have been equivalent to test if a scalar function is positive, instead of computing the eigenvalues of a 2×2 matrix. The reduced Energy-Momentum method appeared originally in [22]. As an illustration, the stability of many relative equilibria for the heavy top are studied in [7] using the reduced Energy-Momentum method.

Another line of research consists in dropping the main assumptions of freeness and compactness. From the geometric point of view, this is equivalent to consider singularities of the momentum map, which creates problems for studying the reduced spaces, since they are not smooth manifolds anymore. The study of singular reduction started in the early 90's (see [21], [1], [5] and [16]) and it is still an active research topic. The extension of the Energy-Momentum method to incorporate relative equilibria for non-free Hamiltonian actions was done in [8] and [15]. For simple mechanical systems, a generalization of the reduced Energy-Momentum method for non-free actions was obtained in [20].

Some topics under current investigation, not mentioned in these notes, are the important areas of persistence and bifurcations of Hamiltonian relative equilibria. This subject starts with an observation of Arnold, ([2]) who proved that non-degenerate relative equilibria for free Hamiltonian actions produce branches of relative equilibria. This means if x is a non-degenerate relative equilibrium with momentum μ, any small enough neighborhood of x contains a relative equilibrium for any value of the momentum close to μ. Later, it was shown in [18] that these persisting branch of relative equilibria form a local symplectic submanifold of the original space. The generalization of these and other results to non-free Hamiltonian actions as well as the development of a systematic Hamiltonian bifurcation theory is still an unfinished topic. The reader can get an idea of the actual status of the subject by looking for instance to [3], [4], [8], [14], [12] and [19].

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